Kinematics and Numerical Algebraic Geometry

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Credits

- Long-Time Collaborator:
  - Andrew Sommese, Univ. of Notre Dame
- “Bertini” Team
  - A.S. & C.W. and…
  - Dan Bates, Colorado State Univ.
  - Jon Hauenstein, North Carolina State Univ.

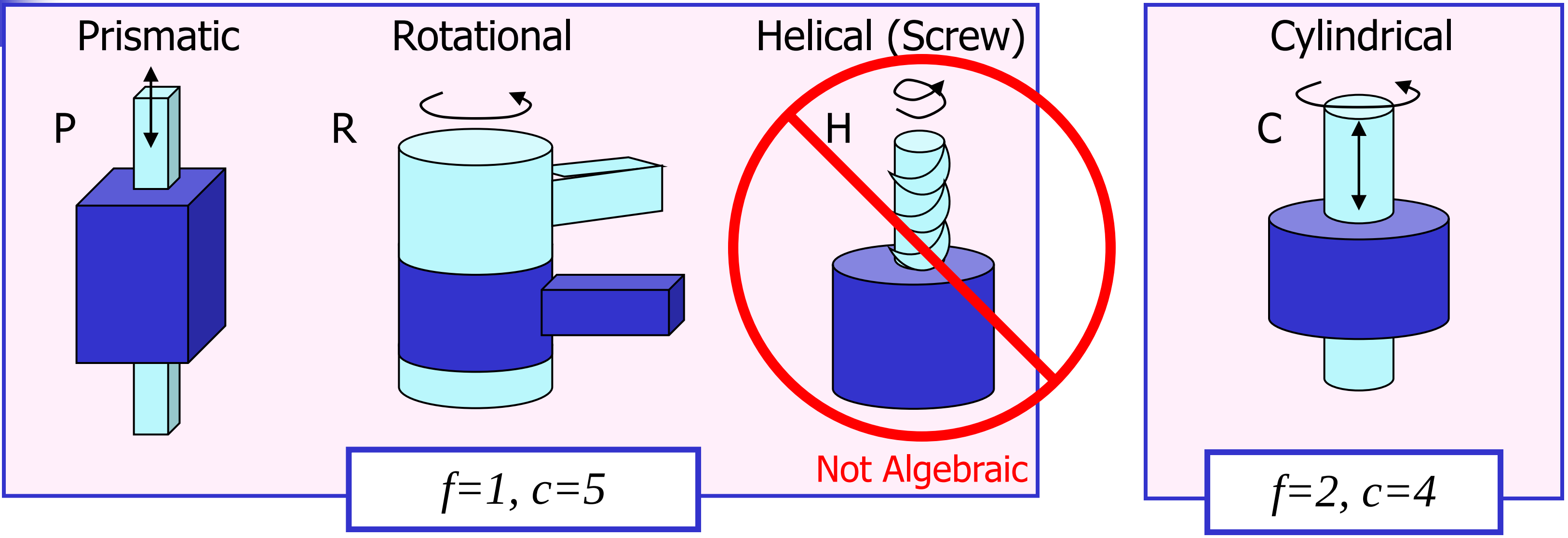
Outline

- Algebraic Kinematics
  - Why most of kinematics is algebraic
  - Kinematics in a nutshell
- Solving polynomial systems
  - Basic polynomial continuation
    - Finding isolated roots
  - Numerical algebraic geometry
    - Dealing with positive-dimensional sets
  - Bertini software package
- Examples from kinematics
- Short Bertini tutorial

Algebraic Kinematics

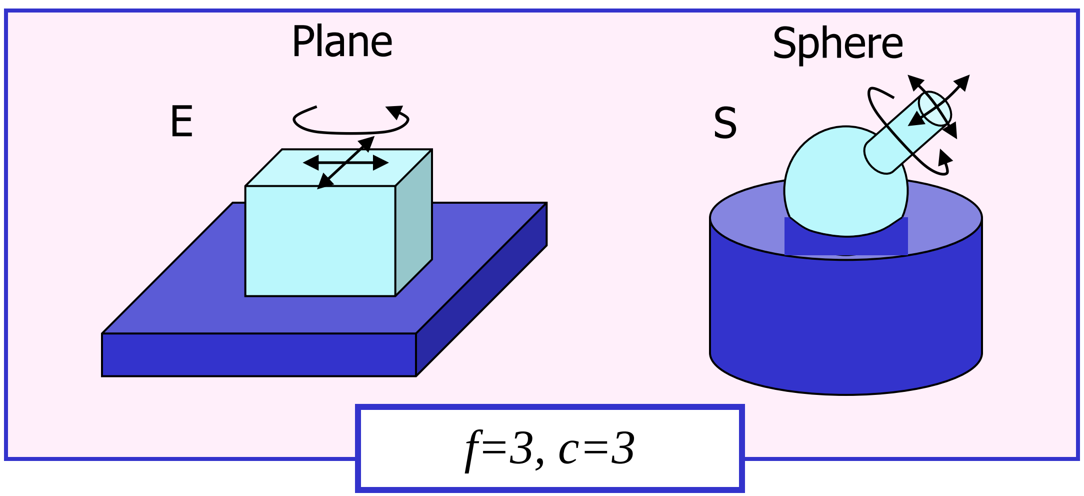
- Rigid-body motions form an algebraic set, *SE(3)*
  - $SE(3) = \{(p,A): p \in \mathbb{R}^3,\ A \in \mathbb{R}^{3\times 3},\ A^T A = I,\ \det A = 1\}$
  - Alternative: Study coordinates, subject to the Study quadric
- The most common joints impose algebraic constraints
- Distance (squared) is also polynomial
  - Cable & tensegrity structures
- $\therefore$ Rigid links + algebraic joints implies *algebraic kinematics*
- Notes:
  - Not all devices have algebraic kinematics:
    - Cams, rolling contact, helical joints
  - Even if not, an algebraic approximation may be quite useful
    - Compliant mechanisms (pseudo-rigid-body model)
  - Most robots, esp. industrial ones, have algebraic kinematics
  - Molecules (incl. proteins) governed by inter-atomic distance constraints have algebraic kinematics

Joints: Lower-order pairs

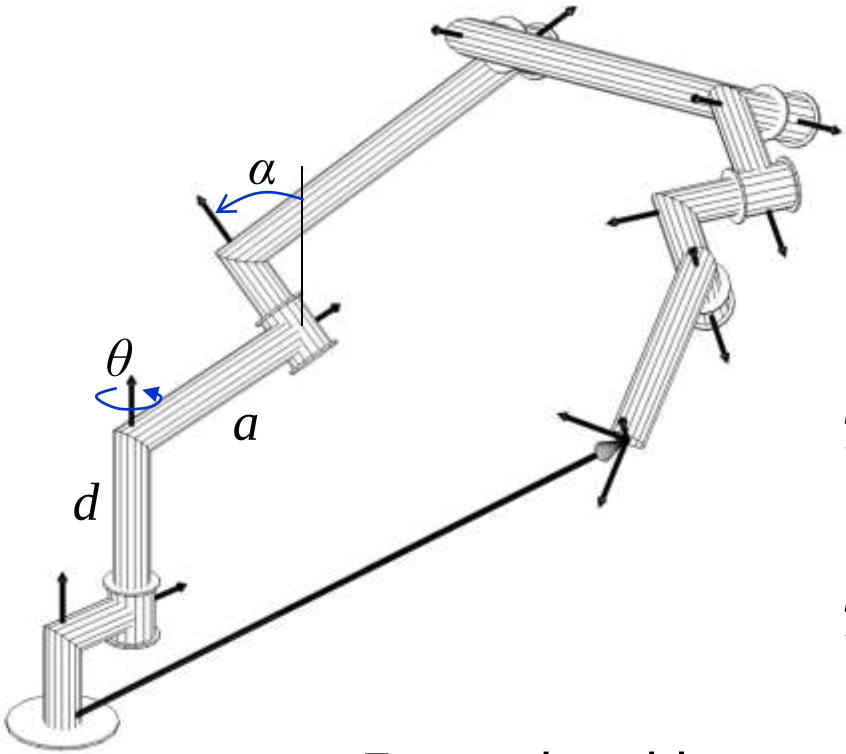


f = freedom

c = constraint in SE(3)



Example: Serial 6R Robot



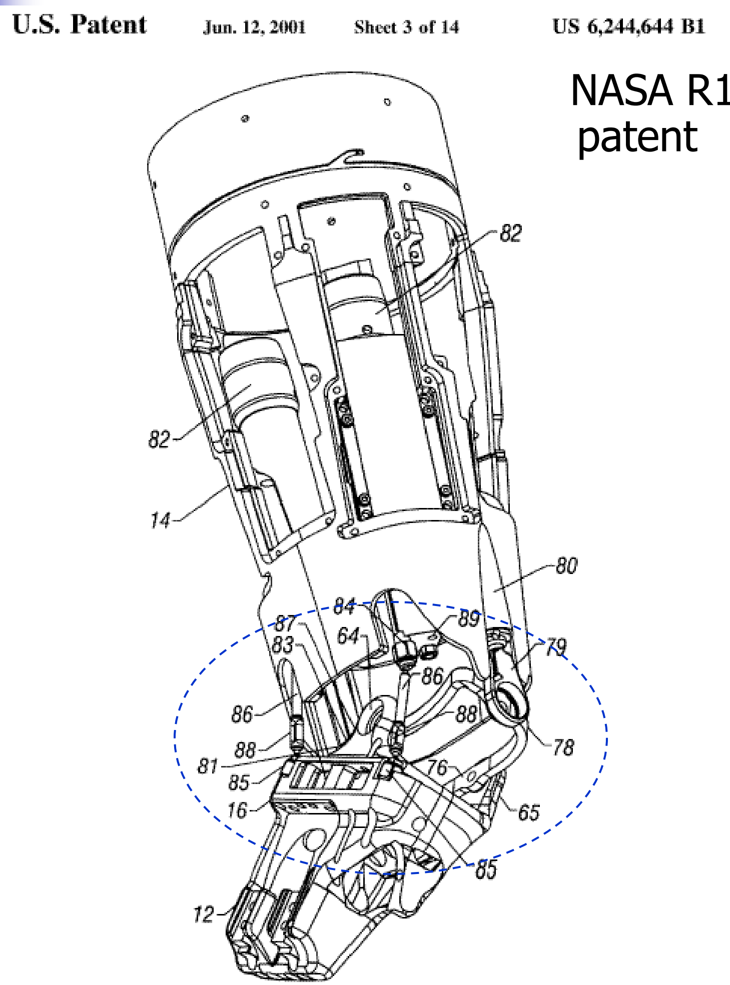
- Parameters given:
  - Length a_i, offset d_i, twist α_i
- Input:
  - Rotation angle at each joint, θ_i
- Output:
  - Position & orientation of end of arm, T_{end}

$$T_{end} = T_1 \cdot T_2 \cdot T_3 \cdot T_4 \cdot T_5 \cdot T_6$$

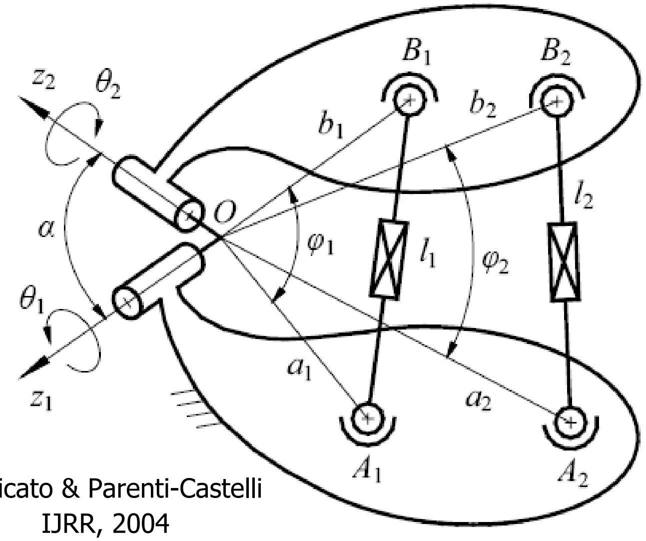
$$T_i = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Forward problem:
  - Unique answer
- Inverse problem:
  - Up to 16 solutions ← Lung-Wen Tsai & Alec Morgan ASME Melville Medal, 1985

Example: Parallel Wrists



NASA R1 patent



Carricato & Parenti-Castelli
IJRR, 2004

Fig. 1. A 2-DoF fully parallel wrist of general geometry.

almost

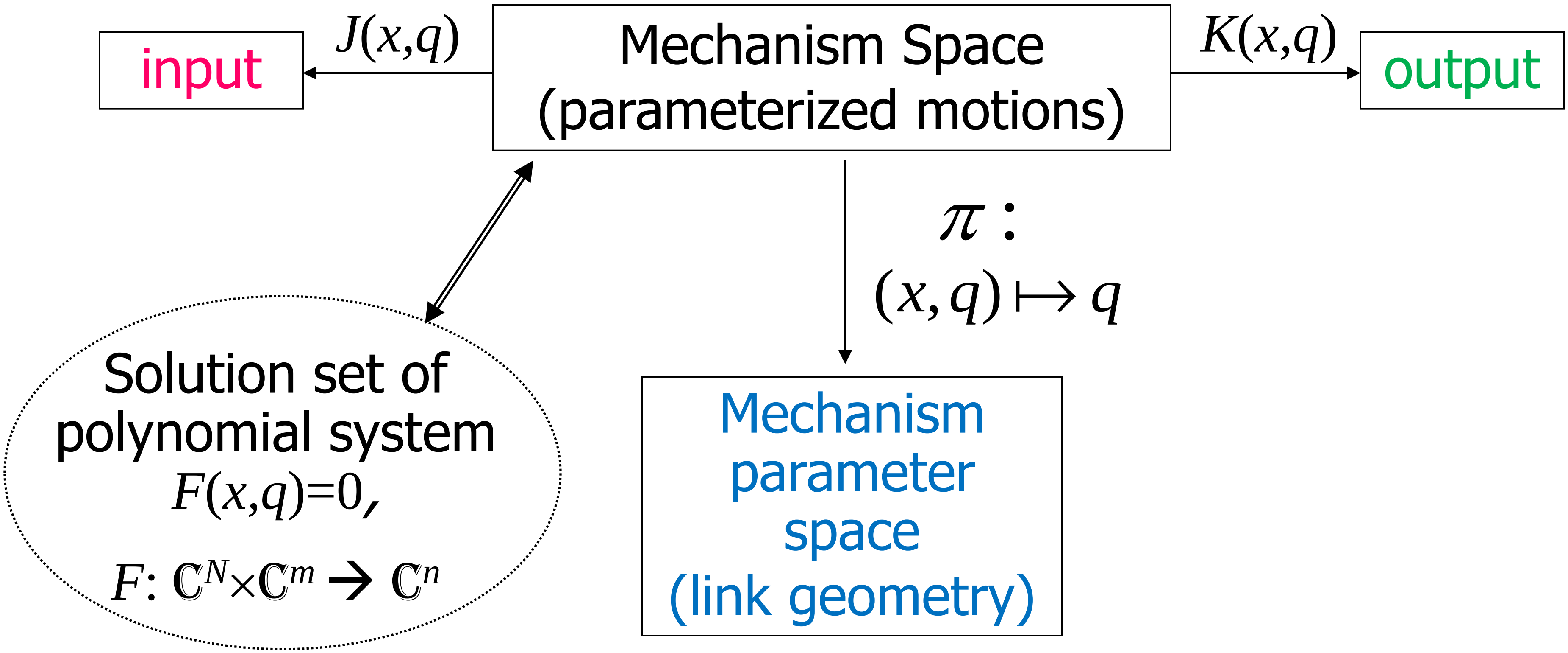
Input: sliders L_1, L_2

Output: angles θ_1, θ_2

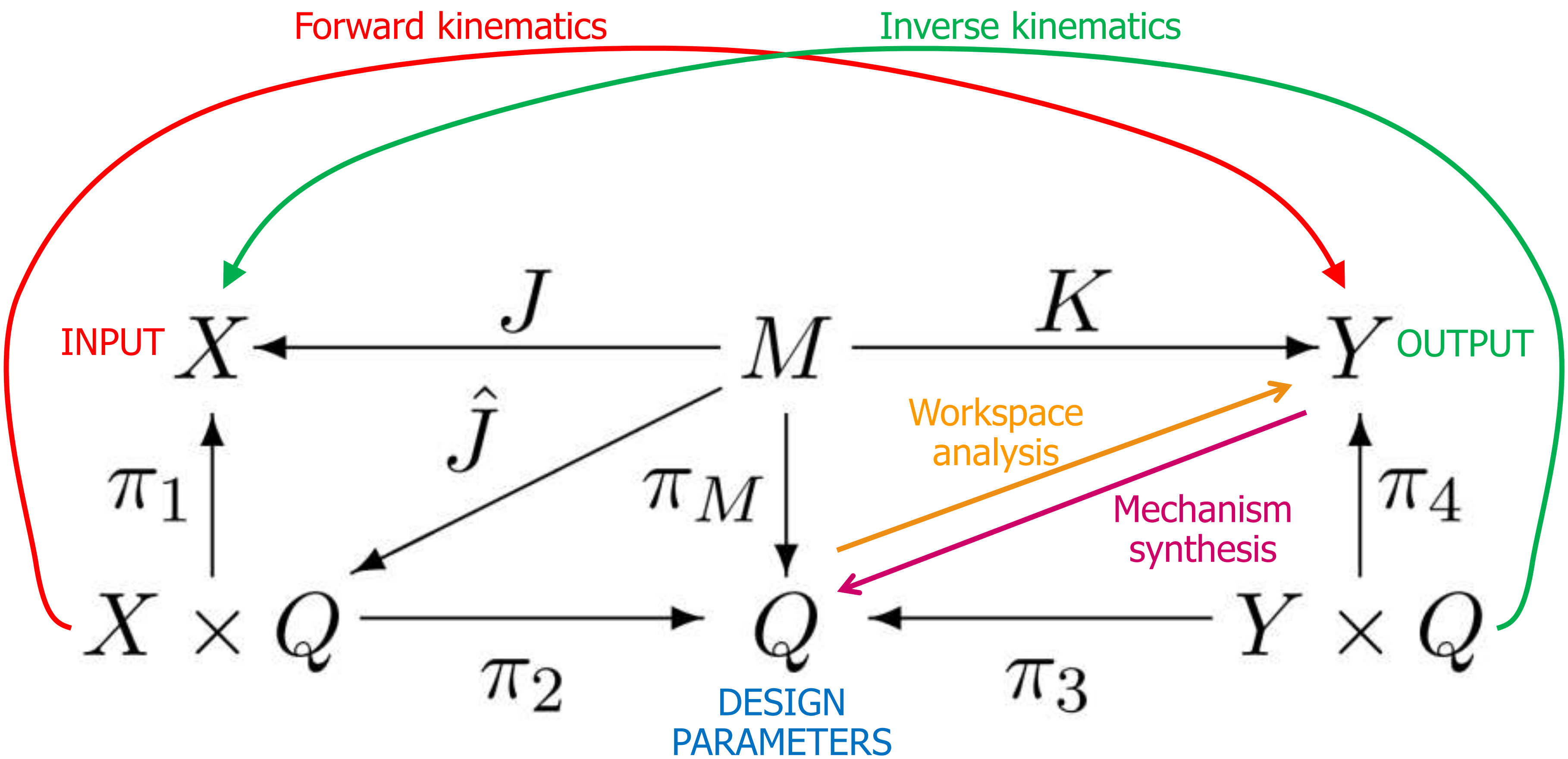
- Inverse problem: 2x2=4 solutions
- Forward problem: 8 solutions

GM

Big Picture


input
$J(x,q)$
Mechanism Space
(parameterized motions)
$K(x,q)$
output
$\pi :$
$(x,q) \mapsto q$
Solution set of
polynomial system
$F(x,q)=0,$
$F: \mathbb{C}^N \times \mathbb{C}^m \rightarrow \mathbb{C}^n$
Mechanism
parameter
space
(link geometry)

Big Picture



Forward kinematics
Inverse kinematics
INPUT
X
J
M
K
Y
OUTPUT
Ĵ
π1
πM
π4
Workspace analysis
Mechanism synthesis
X × Q
Q
Y × Q
π2
π3
DESIGN PARAMETERS

Solving kinematic equations

- Traditional Elimination
  (19th Century)
  - Sylvester resultant, dyalitic elimination
  - Advantages
    - Often runs fastest for small problems
  - Disadvantages
    - Hard to derive, esp. for big problems
    - Numerical stability

- Computer algebra
  (20th-21st Century)
  - Grobner bases, Kronecker elimination
  - Advantages
    - Automated
    - Exact for integer or rational coefficients
  - Disadvantages
    - Cannot handle large systems with real parameters
    - Not easily parallelizable
    - Numerical stability of the final solution step
    - Gives equations not solutions

- Numerical algebraic geometry
  (20th-21st Century)
  - Polynomial continuation
  - Advantages
    - Automated
    - **Parallelizable** – make full use of multinode, multicore processors
    - Can handle large systems with real parameters
    - Robust to special cases
  - Disadvantages
    - Slower on small problems
    - Reliable results but not mathematical proof
    - Gives solutions, not equations

Manfred: "It is essential to use geometric & algebraic pre-processing before applying Grobner or numeric methods"

My addendum: "In engineering, almost always one seeks a numeric answer; at question is how soon to go numeric."

Part II

- Basic polynomial continuation
  - Finding isolated solution points

Basic Total-degree Homotopy

To find all isolated solutions to the polynomial system $F = \{f_1,\ldots,f_N\}$:

Target system

Start sys.

$$H(x,t) = (1-t)\begin{bmatrix} f_1(x_1,\ldots,x_N) \\ \vdots \\ f_N(x_1,\ldots,x_N) \end{bmatrix} + \gamma t \begin{bmatrix} x_1^{d_1} - 1 \\ \vdots \\ x_N^{d_N} - 1 \end{bmatrix} = 0$$

$$d_i = \deg(f_i)$$

γ random, complex.

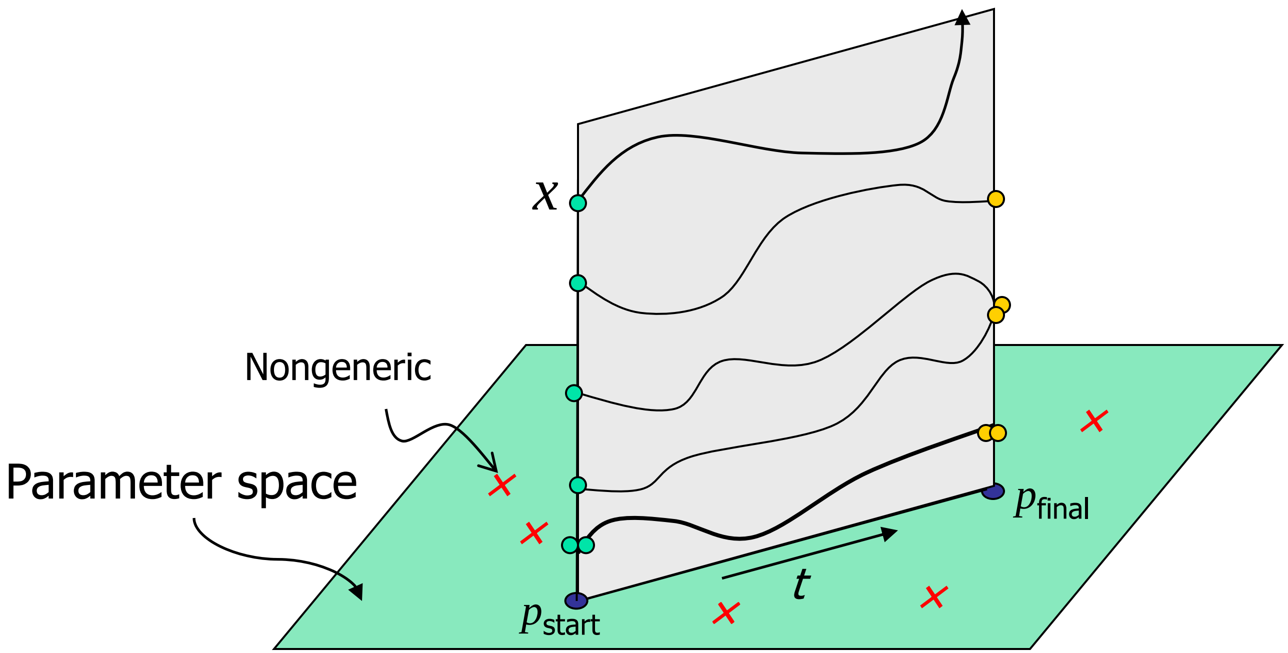
"Probability one" algorithm

Number of paths to track = $d_1 \cdot d_2 \cdots d_N$

Solution paths

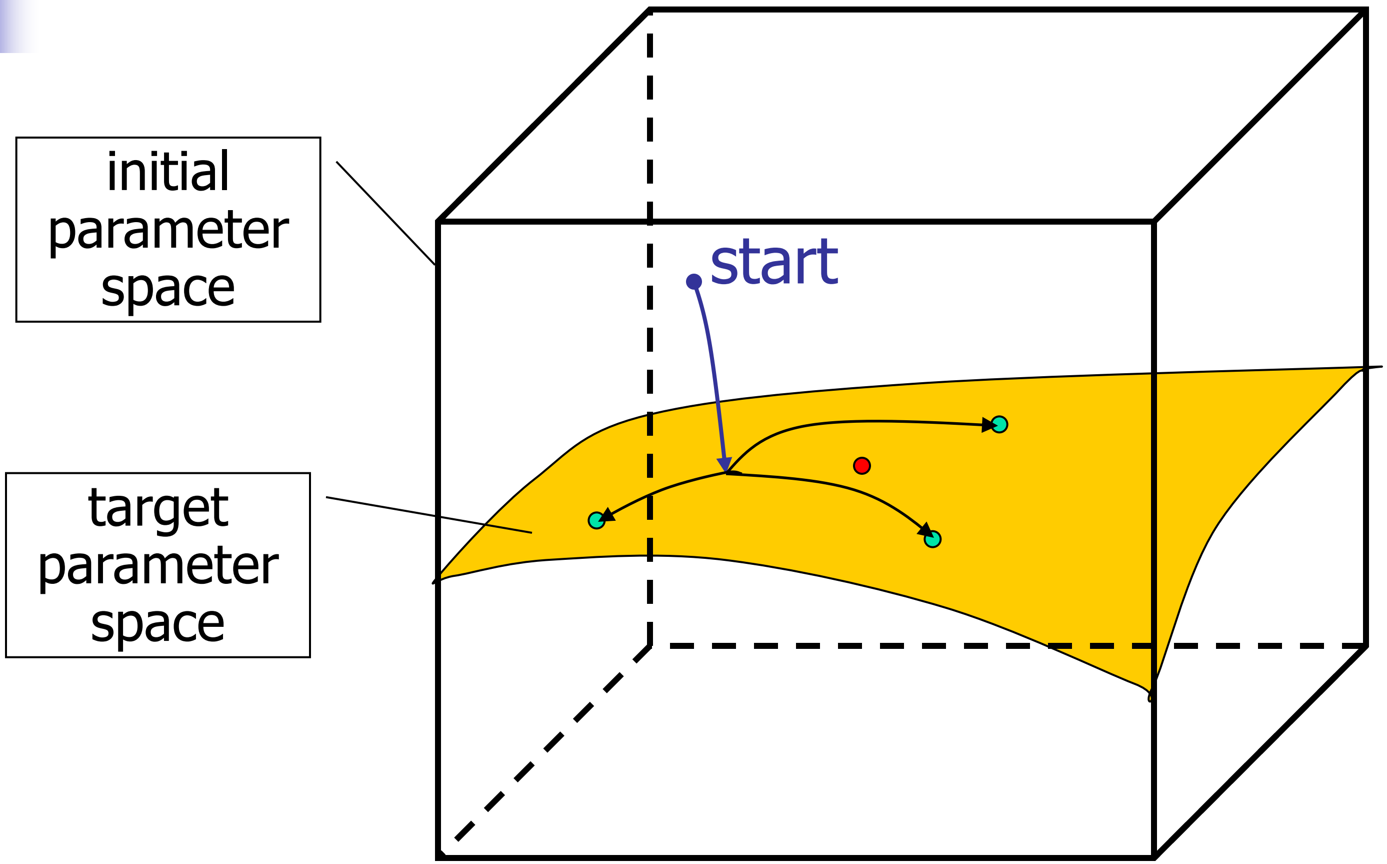
- Paths $x(t)$ implicitly defined by homotopy

$$H\,(x;\,p(t)) = 0$$



Parallelizable: each path can be tracked on a different CPU.

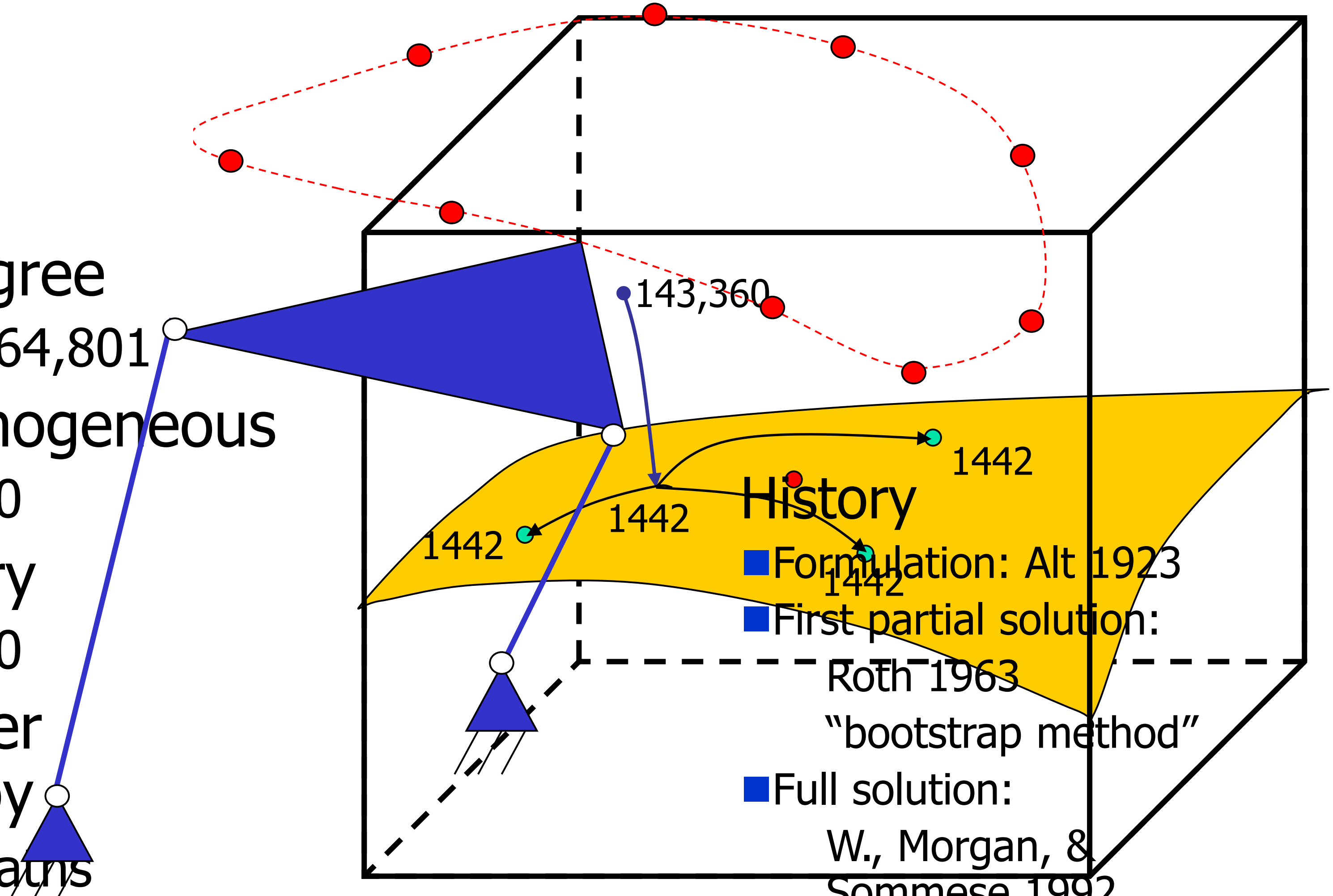
Parameter Continuation



- Start system easy in initial parameter space
- Root count may be much lower in target parameter space
- Initial run is 1-time investment for cheaper target runs

Parameter Continuation: 9-pt path synthesis

- Total degree
  - 7^8=5,764,801
- Multihomogeneous
  - 286,720
- Symmetry
  - 143,360
- Parameter homotopy
  - 1442 paths



History

- Formulation: Alt 1923
- First partial solution: Roth 1963 "bootstrap method"
- Full solution: W., Morgan, & Sommese 1992

Part III

- Numerical Algebraic Geometry
  - Finding & manipulating algebraic sets (points, curves, surfaces,…)

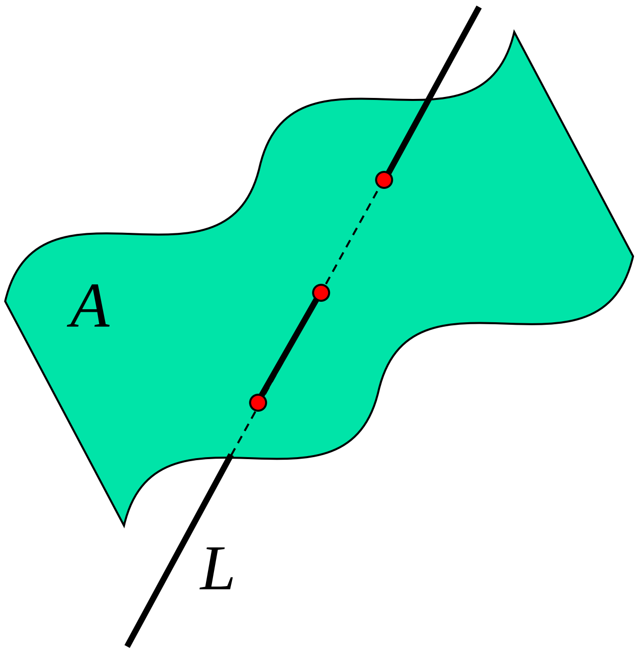
Irreducible Decomposition

Univariate Polynomial	Multivariate System
1 equation, 1 variable	N equations, n variables
Solution points	Irreducible components
Double roots, etc.	Sets with multiplicity
Factorization $c\prod_i (x-a_i)^{\mu_i}$	Irreducible decomposition

Numerical Representation	
List of points	List of witness sets

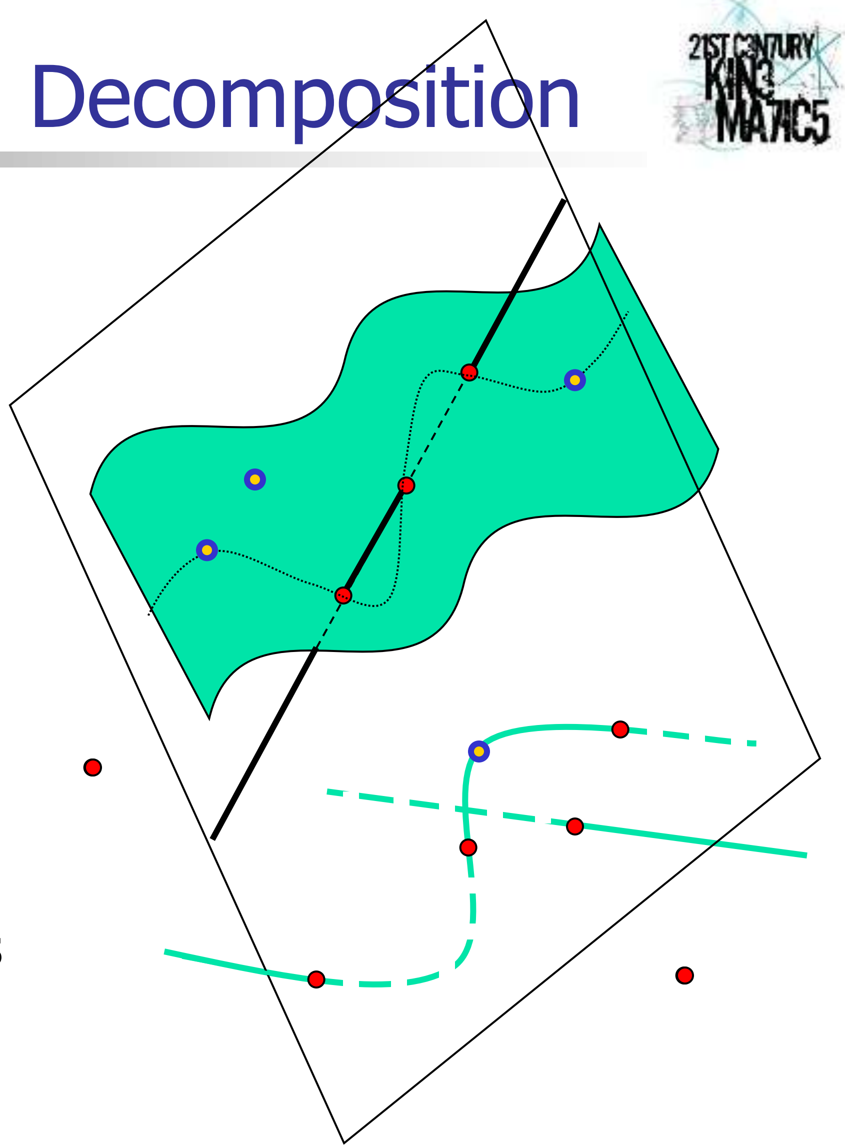
Basic Construct: Witness Set

- Witness set for irreducible algebraic set A is $\{F,L,L\cap A\}$
  - F is a polynomial system such that A is an irreducible component of $V(F)$
  - L is a generic linear space of complementary dimension to A
  - $L\cap A$ is the witness point set
    - *d* points on a degree *d* component



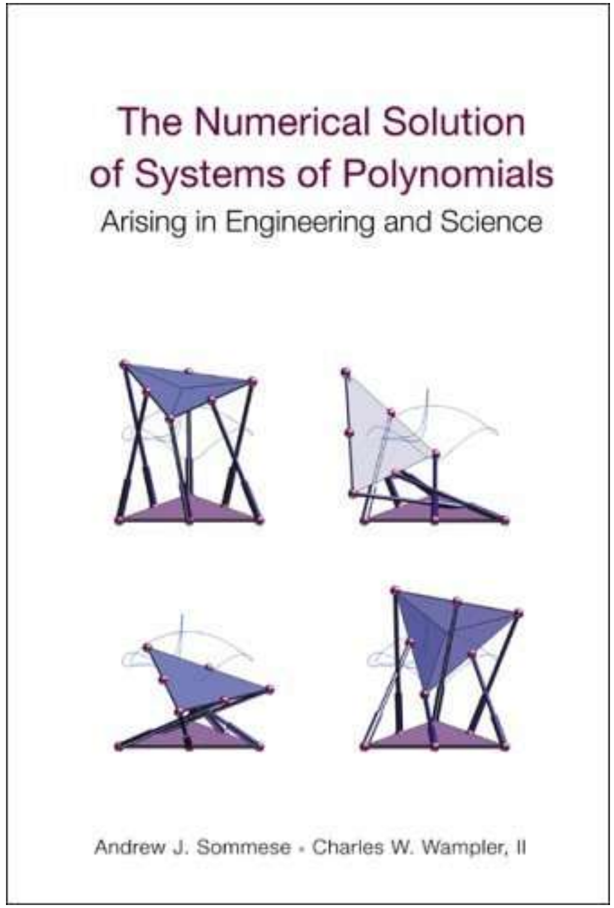
Numerical Irreducible Decomposition

- Witness superset generation
  - Work dimension-by-dimension
  - Slice for every dimension
  - Homotopy finds all isolated solutions at each dimension



- Decomposition
  - Remove “junk” points
    - This gives witness sets by dimension
  - At each dimension, sort witness set into irreducible components
    - This gives the “Numerical Irreducible Decomposition”

For more…



World Scientific, 2005

Software

Ours

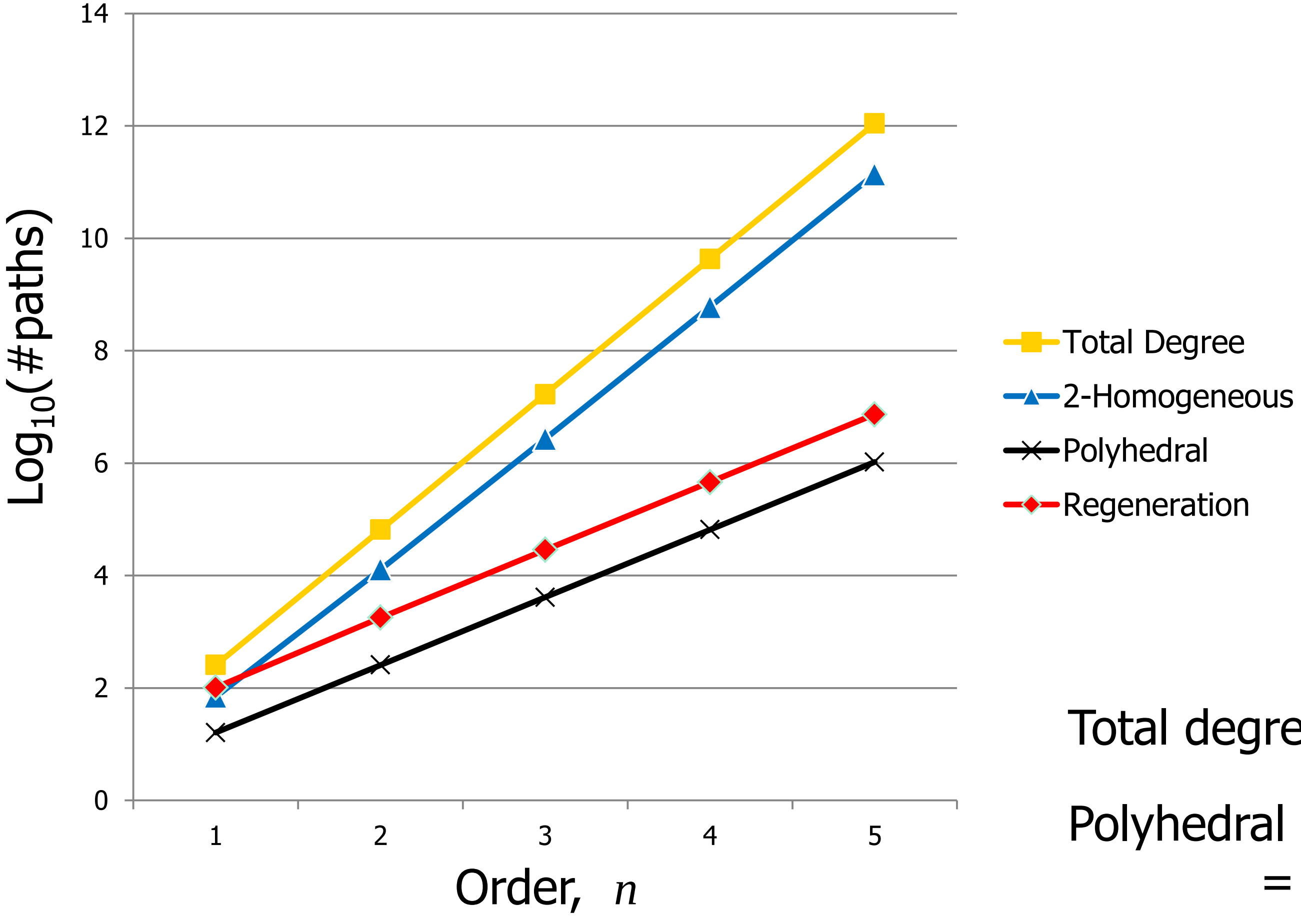
- Bertini (v1.3)
  - Numerical algebraic geometry
    - Robust adaptive multiprecision
    - Deflation of sets with multiplicity>1
    - Regeneration
    - Parallel computing option
  - Authors:
    - Bates, Hauenstein, Sommese & W.
- LocalDimFinder
  - Local dimension test
  - Authors:
    - Hauenstein, Sommese & Wampler
- Free downloads at
  - www.nd.edu/~sommese/bertini/

Others

- Hom4PS (v2.0)
  - Isolated solutions only
  - Fast polyhedral
  - Author: T.-Y. Li (MSU)
- PHC
  - Numerical algebraic geometry
  - Polyhedral method
  - Author: Jan Verschelde (UIC)
- POLSYS_PLP, POLSYS_GLP
  - Isolated solutions only
  - Linear product homotopies
  - Author: Layne Watson (VaTech)

Test Run: Lotka-Volterra Systems

- Discretized PDE (finite differences) population model
  - Order n system has $8n$ sparse bilinear equations



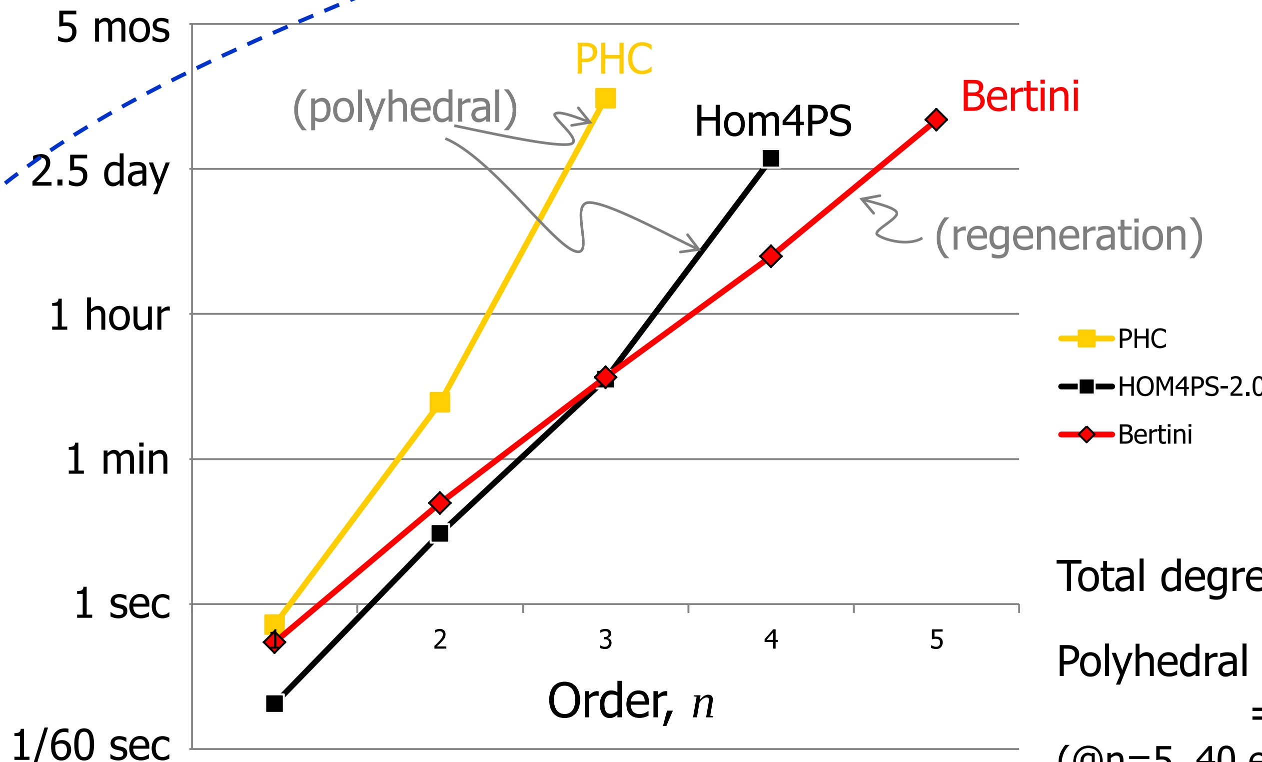
Total degree = 2^{8n}

Polyhedral (mixed volume) = 2^{4n} is exact

Test Run: Lotka-Volterra PDE Systems

- Order n system has $8n$ sparse bilinear equations
- Time Summary -- Single Processor

Credit: Jon Hauenstein 2009



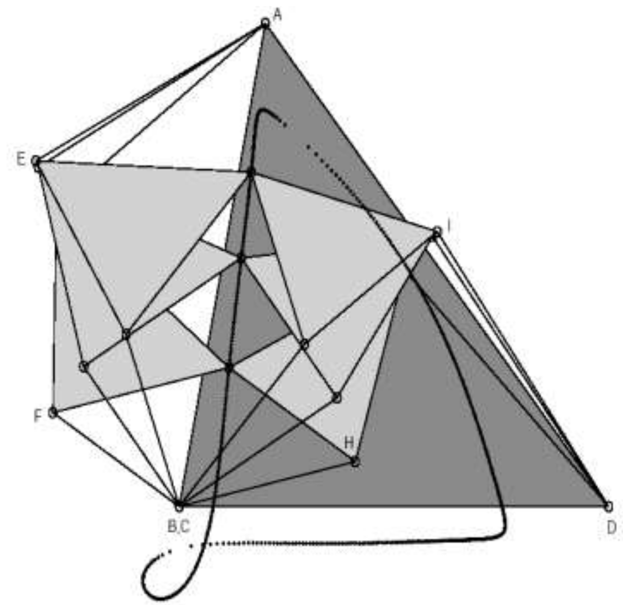
Total degree = 2^{8n}

Polyhedral (mixed volume) = 2^{4n} is exact (@n=5, 40 eqs, ~10^6 roots)

- Regeneration parallelizes easily (polyhedral does not)

Part IV: Examples

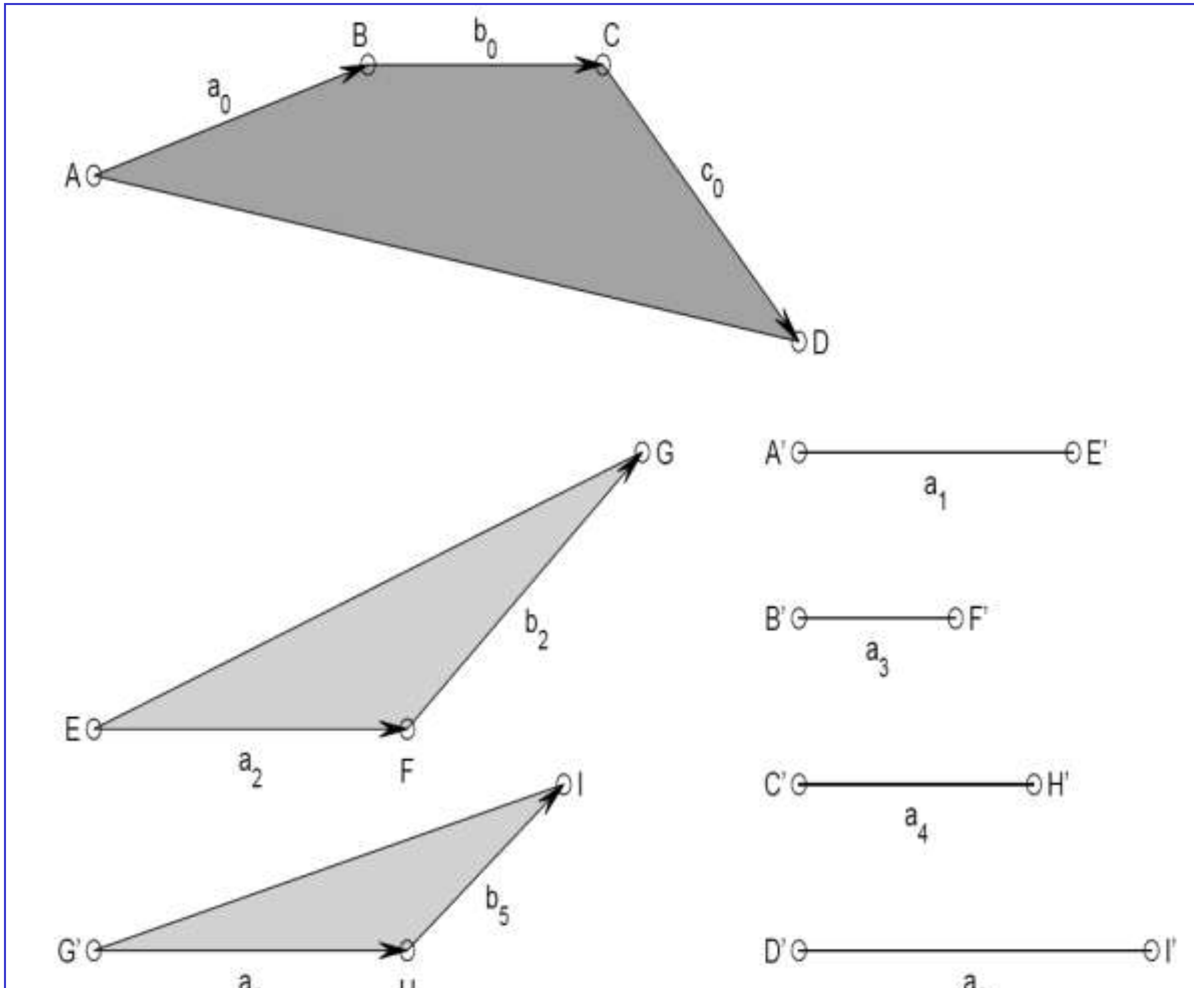
- Let's see Numerical Algebraic Geometry at work in kinematics



Example: 7-bar Structure

Problem:

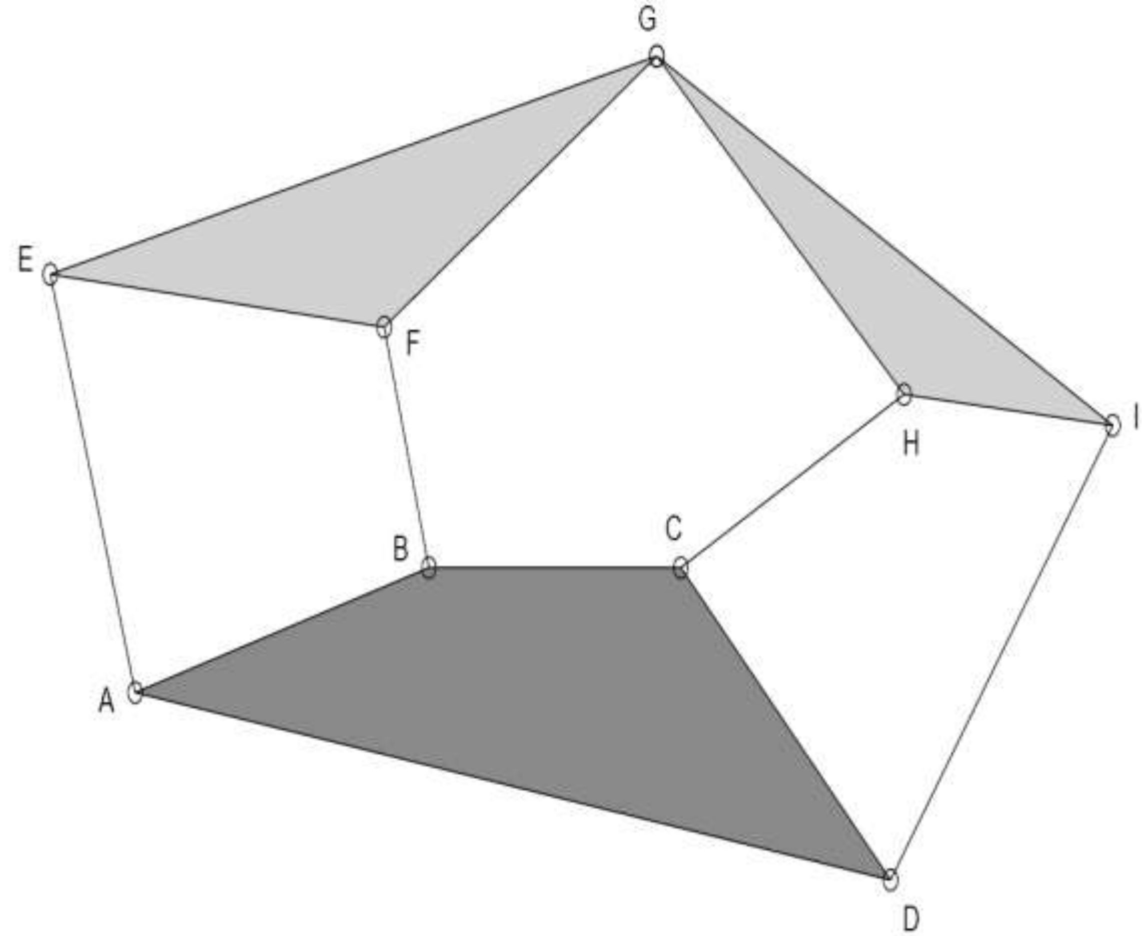
Assemble these 7 pieces, as labeled.



Result for Generic Links

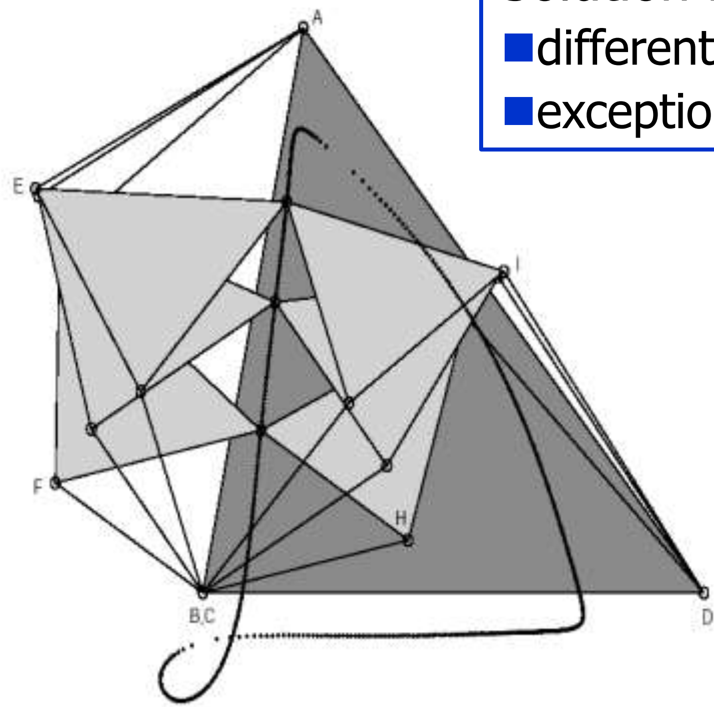
18 rigid structures

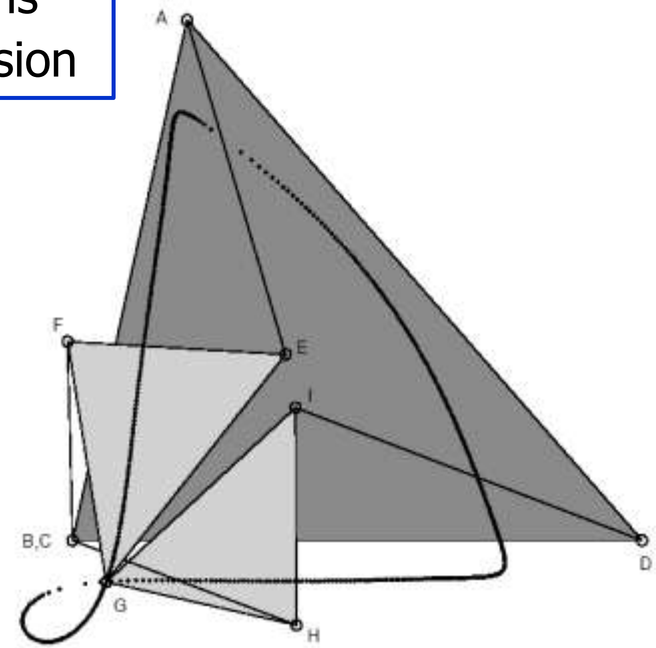
- 8 real, 10 complex for this set of links.
- All isolated – can be found with traditional homotopy



Special Links (Roberts Cognates)

Solution Properties:
- different dimensions
- exceptional dimension





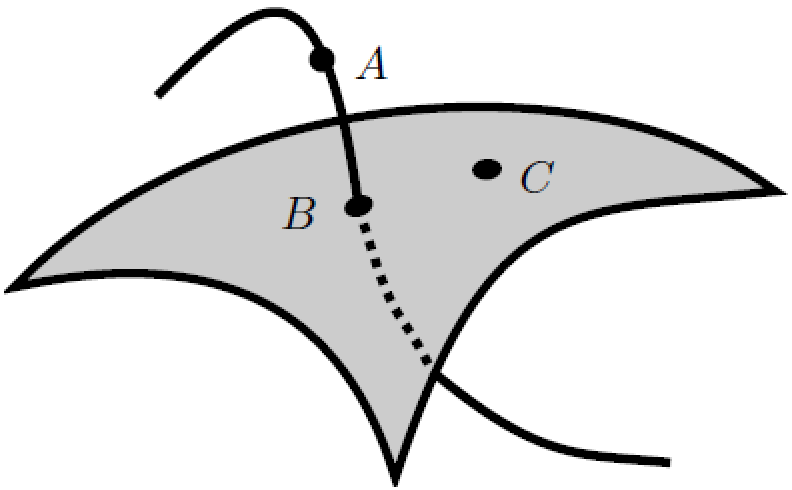
Dimension 1:

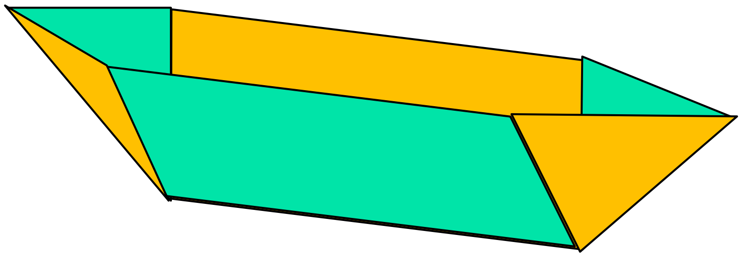
6^{th} degree four-bar motion

Dimension 0:

1 of 6 isolated (rigid) assemblies

“Kinematotropic” mechanisms





“Boat” 6R mechanism

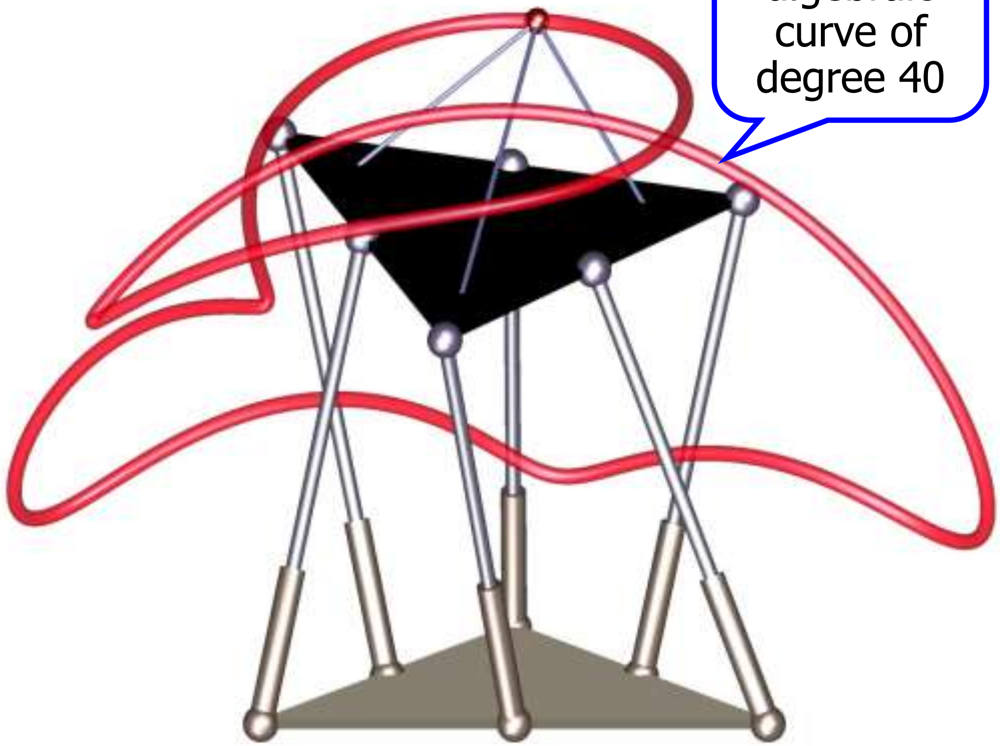
Solution Properties:

- Curve and surface that meet

Exceptional Stewart-Gough Platform

Griffis-Duffy platforms

- Case 1: Top & bottom plates are equilateral triangles
  - Degree of top platform motion in Study (dual quaternion) coordinates is 28
  - Degree of path of a tracing point is 40.
- Case 2: In addition, leg lengths equal & plates congruent
  - Factors as 6+(6+6+6)+4=28



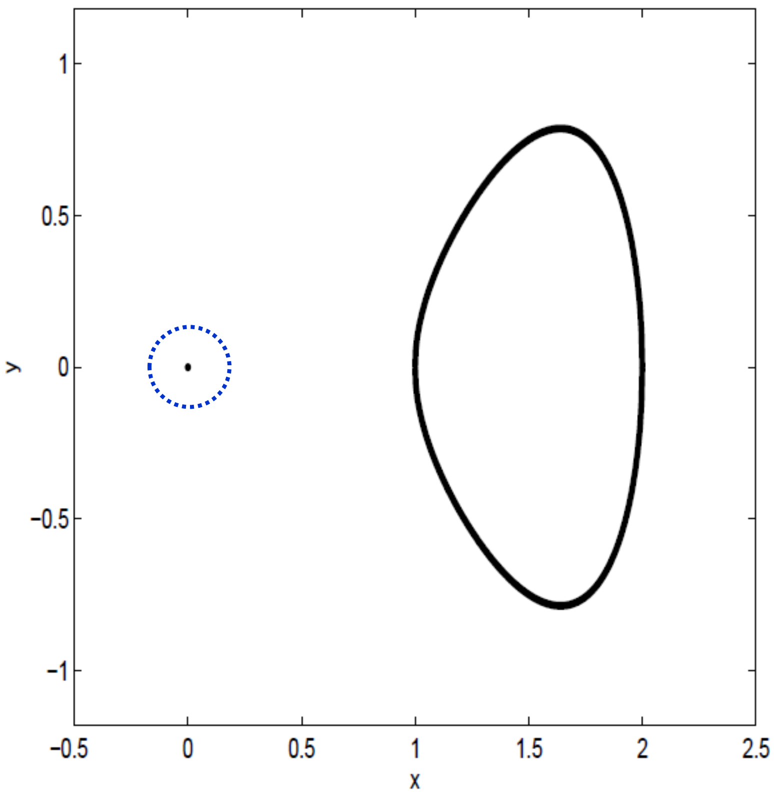
This is an algebraic curve of degree 40

Even More Exceptional Stewart-Gough Platform

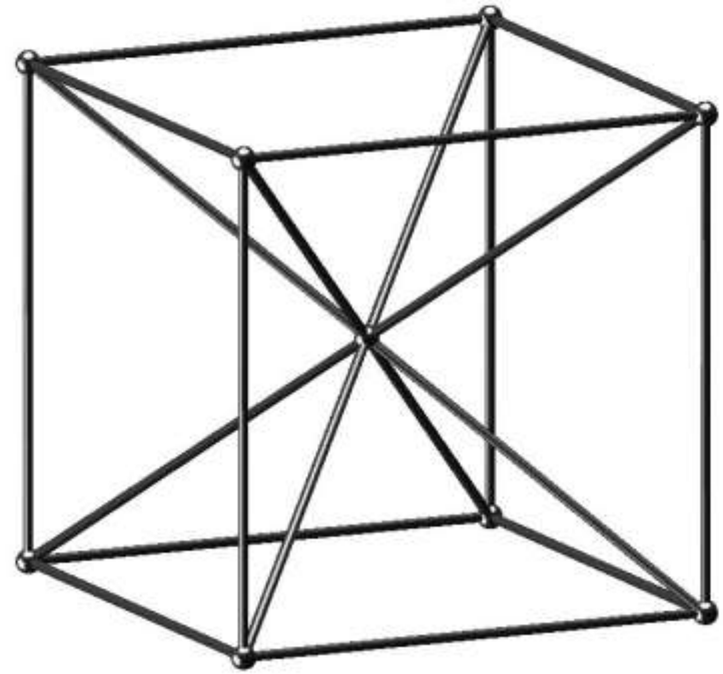
- As before, but with
  - leg lengths = altitude of base triangle
  - "Foldable Griffis-Duffy Platform"
- Degree 28 component now factors as
  - $3\times[2\times1]+3\times2+4+(4+4+4)$
  - We have extracted the *real* parts of these complex components
    - 3 double lines, 3 quadrics, 1 quartic

Solution Properties:
- exceptional dimension
- sets w/ multiplicity = 2

Real vs. Complex Dimension



The real solutions of
$y^2 + x^2(x-1)(x-2)$

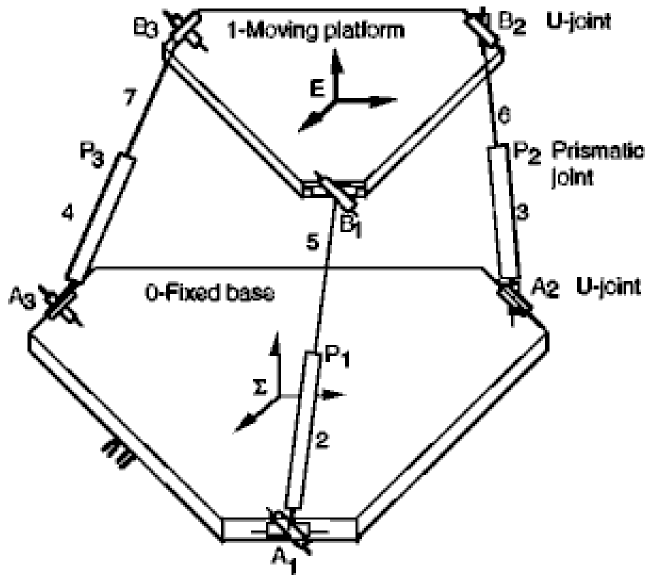


12-bar mechanism

Solution Properties:

- Isolated real point on a complex curve (double root)
- 2[nd] component is a real curve

Seoul Nat'l Univ. 3-UPU mechanism



Tsai 3-UPU
Translational platform
1996

SNU 3-UPU
Frank Park
2001
(joints intersect)

Solution Properties:

- Pose shown is isolated but multiplicity = 4

Part V

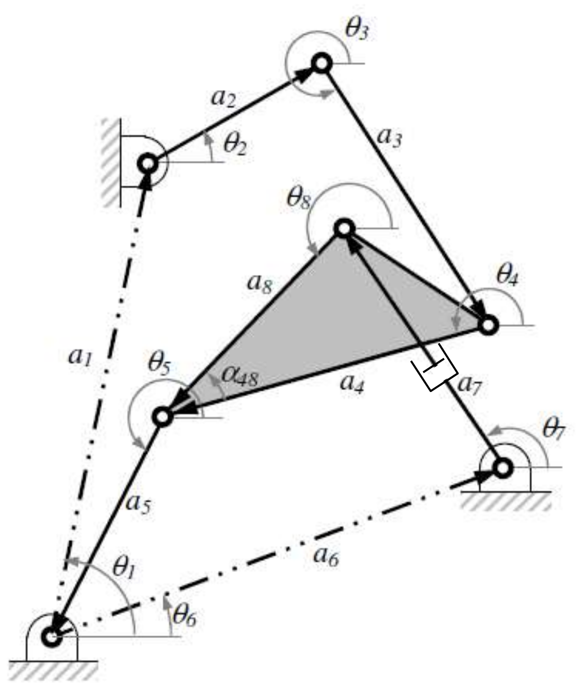
- What's next?
- Wrap-up
- Bertini demos

What's Next?

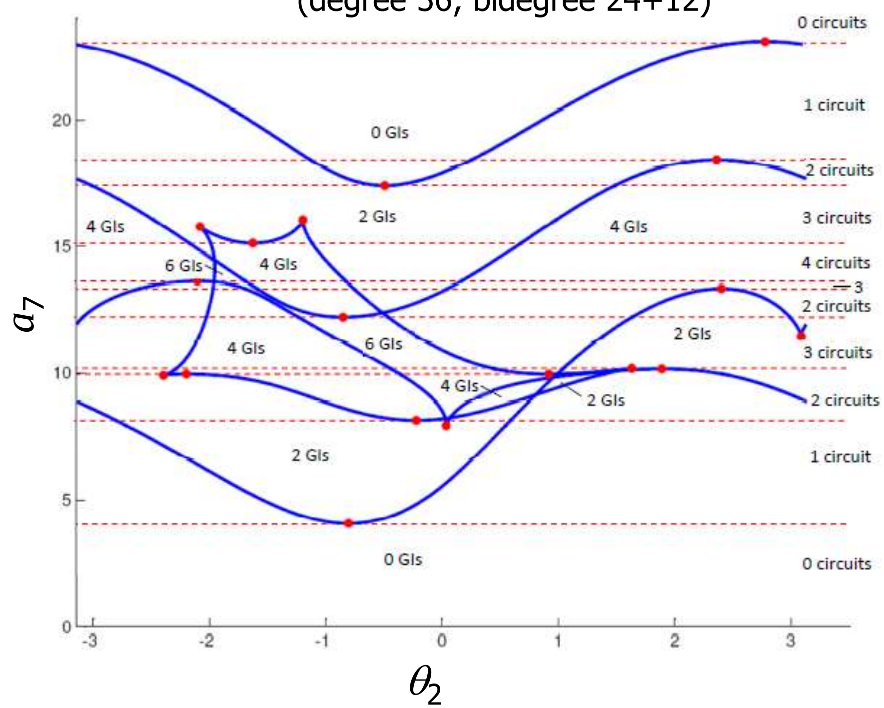
- Bertini book in progress
  - (More users' manual than S&W 2005 monograph)
- Bertini open-source release planned
  - (Only executables are available at present)
- Decomposition of **real** sets
  - Currently, Bertini solves in complex space
    - Fine for nonsingular isolated solutions
    - But for singular points and for real curves, surfaces, etc.,
      - Real sets can start and stop (i.e., turn, fold, etc.)
      - Need to form & solve conditions for where the real points lie inside the complex solution sets
    - Algorithms for curves & surfaces have been developed, but not yet released in Bertini

Real Surface = 2DOF Motion

- Stephenson III w/ prismatic joint added
  - Inputs: θ_2, a_7
  - Myszka, Murray, Wampler: IDETC, Wed., 11:30am



Turning Point Curve
(degree 36; bidegree 24+12)



Wrap-up

- Much of kinematics is applied algebraic geometry
  - Mechanism space formulation of kinematic problems
  - Mechanisms may have:
    - Solutions at different dimensions
    - Higher multiplicity
    - Real dimension different than complex dimension
- Numerical Algebraic Geometry
  - Finds isolated solution points and positive-dimensional solutions
    - "Numerical Irreducible Decomposition"
    - Based on polynomial continuation for finding isolated points
  - Advances in methodology
    - Eqn.-by-Eqn. methods for large systems (Regeneration)
    - Deflation of multiplicities
    - Adaptive multiple precision
    - Parallel computing
  - Bertini v1.3 offers all this & more
- A 21st Century kinematician needs 21st Century tools!
- Bertini demo to follow…