Classification of RRSS linkages

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Abstract

This paper classifies the movement of the RRSS spatial linkage in terms of its link dimensions. The constraint equation for this chain relates the rotation angles of the input crank and coupler around the two revolute joints. The condition for a real valued solution for the coupler angle as a function of the input angle yields a quartic polynomial. We use three polynomial discriminants obtained from this quartic to find explicit conditions for the cases available for the movement of the input crank. A similar analysis characterizes the coupler movement by determining conditions that ensure real solutions for the input angle as a function of the coupler angle. Combining the results we obtain parameters that characterize movement of the RRSS linkage. Examples for the various cases are provided.

1. Introduction

This paper studies the relationship between the physical dimensions of a spatial RRSS linkage and the range of movement about the revolute joints of its input crank and coupler link—R denotes a revolute, or hinged, joint and S denotes a spherical, or ball, joint. The goal is to obtain a general description of the motion of this linkage simply from its link dimensions. The well-known Grashof’s condition for planar 4R linkages is an example of the relationship that we seeking [3,12]. Our approach follows that presented by Murray and Larochelle [5] for planar and spherical 4R and spatial RCCC linkages. Key to this approach is determining the limit positions of crank rotation about the revolute joints. A planar 4R linkage that has a rotatable crank is often called a...
“Grashof” linkage. It is convenient in what follows to maintain this terminology and refer to an RRSS linkage with a rotatable crank as “Grashof” also.

The RRSS linkage (Fig. 1) is an inversion of the RSSR linkage, and the classification strategy for the two is essentially identical—our focus on the RRSS linkage arises from its use as a spatial motion generator. Nolle [6] performed the limit analysis of the RGGR linkage (G is another symbol for an S-joint) by factoring the general quartic into two quadratics. Sticher [9] used an “Ellipse Diagram” to visualize the various cases of the RSSR linkage. Williams and Reinholtz [13] formulated conditions on the input crank angle that ensure closure of RSSR and RRSS linkages using polynomial discriminants. Our approach in [10] focussed on the input angle constraints that guaranteed an output angle solution, in which we determined the special cases of the associated discriminants.

A more detailed analysis is obtained by considering conditions on the rotation of links about both R-joints. Ting and Dou [11] analyzed the limits to movement about both R-joints and obtained two sets of discriminant parameters similar to Williams and Reinholtz. The result was a classification of these linkages based on the number of branches of the input/output function. Rastegar and Tu [8] used geometric approximations to the closure of the RSSR linkage to obtain conditions on rotation about both joints to obtain approximate “Grashof-type” conditions. In this paper we use a set of discriminants obtained from solutions to the closure equations for both the input and output links, and obtain four parameters for each of the revolute joints that define the relative rotatability of the attached links. The signs of these parameters provide a classification of the various types of RRSS linkages, including an array of special cases.

2. Position analysis of an RRSS linkage

The RRSS linkage is formed by connecting a floating link to ground using an RR and an SS chain. The resulting linkage can be used to guide a workpiece through three general spatial positions [7].
To analyze this linkage, we introduce a fixed frame $F$. Let $R_1$ and $R_2$ be the axes of the two revolute joints, then the common normal to these lines intersects them in the points $C_1$ and $C_2$, respectively. We choose $C_1$ to be the origin of $F$, and direct its $Z$-axis along $R_1$. We choose the $X$-axis such that the point $S_1$ that defines the fixed S-joint lies in the $XZ$ plane. See Fig. 1. In what follows we denote the $XZ$ plane as $\mathcal{X}$.

For convenience, we also introduce a moving frame $M$, which we attach to the point $C_2$ with its $Z'$-axis along $R_2$. The $X'$-axis is defined so the center of the moving S-joint lies in the $X'Z'$ plane, which we refer to as the plane $\mathcal{P}$.

The position and orientation of $M$ relative to $F$, is defined by the product of $4 \times 4$ homogeneous transforms that define the kinematics equations for the RR chain,

$$[D] = [Z(\theta, 0)] [X(x, a)] [Z(\phi, 0)],$$

(1)

where $[Z(\cdot, \cdot)]$ and $[X(\cdot, \cdot)]$ define the coordinate screw displacements along the $z$ and $x$ axes, respectively [1,4]. The parameters $(x, a)$ define the angle of twist and length of the crank, and $\theta$ and $\phi$ are the input angle and coupler angle, respectively.

The kinematics equations transform the coordinates of the moving S-joint, $s_2 = (b, 0, q)$, to the fixed frame such that $s_2 = [D]s_2$. The trajectory of $s_2$ lies on a sphere about the fixed S-joint $S_1 = (r, 0, p)$. This provides the constraint

$$(S_1 - [D]s_2)^T (S_1 - [D]s_2) = h^2,$$

(2)

where $h$ is the length of the SS link. Expand this to obtain the constraint equation for the RRSS linkage

$$\mathcal{C}(\theta, \phi) : -2rb \cos \theta \cos \phi + 2br \cos x \sin \theta \sin \phi + 2ab \cos \phi - 2hp \sin x \sin \phi - 2ra \cos \theta
- 2qr \sin x \sin \theta + a^2 + b^2 + p^2 + q^2 + r^2 - h^2 - 2pq \cos x = 0.$$  

(3)

For convenience we introduce the constants $A_i, B_i, i = 1, 2$ and $C_i, i = 1, 2, 3$, we have

$$\mathcal{C}(\theta, \phi) : A_1 \cos \theta \cos \phi + B_1 \sin \theta \sin \phi + A_2 \cos \phi + B_2 \sin \phi + C_1 \cos \theta + C_2 \sin \theta + C_3 = 0;$$

(4)

where

$$A_1 = -2br, \quad A_2 = 2ab, \quad B_1 = 2br \cos x,$$

$$B_2 = -2hp \sin x, \quad C_1 = -2ar,$$

$$C_2 = -2qr \sin x, \quad C_3 = a^2 + b^2 - h^2 + p^2 + q^2 + r^2 - 2pq \cos x.$$

(5)

The constraint equation $\mathcal{C}(\theta, \phi)$ defines the relationship between the input angle $\theta$ and the coupler angle $\phi$ that ensures closure of the RRSS linkage. The ranges of movement for both the input and coupler angles are obtained from limit conditions that define the existence of a solution to $\mathcal{C}(\theta, \phi)$ for one of these angles in terms of the other.
2.1. Limit condition for the crank angle

We can group the terms of the constraint equation \( C(h, \phi) \) to obtain an equation for \( \phi \),

\[
A(\theta) \cos \phi + B(\theta) \sin \phi + C(\theta) = 0, \tag{6}
\]

where

\[
A(\theta) = A_1 \cos \theta + A_2 = 2b(a - r \cos \theta),
B(\theta) = B_1 \sin \theta + B_2 = 2b(r \cos \alpha \sin \theta - p \sin \alpha),
C(\theta) = C_1 \cos \theta + C_2 \sin \theta + C_3 = -2ar \cos \theta - 2qr \sin \alpha \sin \theta + a^2 + b^2 - h^2 + p^2 + q^2 + r^2 - 2pq \cos \alpha. \tag{7}
\]

The solution of this Eq. (6) is

\[
\phi(\theta) = \arctan \frac{B}{A} \pm \arccos \frac{C}{\sqrt{A^2 + B^2}}. \tag{8}
\]

It is easy to see that real values exist only if the input angle \( \theta \) satisfies the condition

\[
A_\theta : A^2(\theta) + B^2(\theta) - C^2(\theta) \geq 0. \tag{9}
\]

The solutions to the equation \( A_\theta = 0 \) define the limits to the range of movement of the input crank angle.

2.1.1. Geometric description

It is useful to consider the geometric meaning of the limit condition \( A_\theta = 0 \). Notice that

\[
C^2 = (A \cos \phi + B \sin \phi)^2, \quad \text{therefore} \quad A_\theta \text{ can be written as}
A_\theta : A^2 \sin^2 \phi + B^2 \cos^2 \phi - 2AB \cos \phi \sin \phi. \tag{10}
\]

We can relate this to the geometry of the linkage.

Recall that the \( X'Z' \) plane, \( \mathcal{P} \), of the coupler frame \( M \) is defined by the line \( R_2 \) and point \( S_2 \). Transform the coordinates of the fixed spherical joint \( S_1 \) to \( M \) by computing

\[
s_1 = [D]^{-1} s_1 = \begin{cases}
x_1 \\
y_1 \\
z_1
\end{cases} = \frac{1}{2b} \begin{cases}
-A \cos \phi - B \sin \phi \\
-B \cos \phi + A \sin \phi \\
2b(p \cos \alpha + r \sin \alpha \sin \theta)
\end{cases}.
\tag{11}
\]

Now compute \( y_1^2 \) and compare the result to (10) to obtain the relation

\[
A_\theta = 4b^2 y_1^2. \tag{12}
\]

The result is that \( A_\theta = 0 \) implies that \( y_1 = 0 \), which in turn means that the plane \( \mathcal{P} \) passes through the fixed pivot \( S_1 \).

Thus, we see that the coupler link and the SS chain are aligned when the linkage reaches a limit to the input crank rotation. Lee et al. [2] obtained similar results by considering the plane formed by the moving axis \( R_2 \) and the fixed pivot \( S_1 \).

If \( A_\theta > 0 \), then for each angle \( \theta \), the plane \( \mathcal{P} \) is located such that the fixed joint \( S_1 \) may be in one of two symmetric locations on either side of \( \mathcal{P} \).
2.2. Limit condition for the coupler angle

The terms of the constraint equation can be grouped to determine $\theta$ as a function of $\phi$, that is

$$A(\phi) \cos \theta + B(\phi) \sin \theta + C(\phi) = 0,$$

(13)

where

$$A(\phi) = A_1 \cos \phi + C_1 = -2r(a + b \cos \phi),$$

$$B(\phi) = B_1 \sin \phi + C_2 = 2r(b \cos z \sin \phi - q \sin z),$$

$$C(\phi) = A_2 \cos \phi + B_2 \sin \phi + C_3 = 2ab \cos \phi - 2bp \sin z \sin \phi + a^2 + b^2 - h^2 + p^2 + q^2 + r^2 - 2pq \cos z.$$

The angle $\theta$ is given by

$$\theta(\phi) = \arctan \frac{B}{A} \pm \arccos \frac{C}{\sqrt{A^2 + B^2}}.$$

(15)

As above, the condition for the existence of real solutions is that

$$A_{\phi} : A^2(\phi) + B^2(\phi) - C^2(\phi) \geq 0.$$

(16)

The real roots of $A_{\phi} = 0$ identify limits to the range of movement of the coupler crank angle.

2.2.1. Geometric description

The geometric meaning of the limit condition $A_{\phi} = 0$ can be obtained in a similar manner as above. Compute the coordinates of the spherical joint $S_2$ in $F$, that is

$$S_2 = [D]s_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \frac{1}{2r} \begin{bmatrix} -A \cos \theta - B \sin \theta \\ -B \cos \theta - A \sin \theta \\ 2r(q \cos z + b \sin z \sin \phi) \end{bmatrix}.$$

(17)

Now using the fact that $C^2 = (A \cos \theta + B \sin \theta)^2$, we obtain

$$A_{\phi} = 4r^2 y_2^2.$$

(18)

Thus, the coupler angle reaches its limits when the moving pivot $S_2$ lies in the $XZ$ coordinate plane of $F$.

2.3. The limit polynomials

The equations $A_{\theta} = 0$ and $A_{\phi} = 0$ define the limit positions of the joint rotation for the fixed and moving revolute joints, respectively. We transform the cosine and sine terms in these equations into algebraic form by making the substitutions $x = \tan(\theta/2)$ and $y = \tan(\phi/2)$.

For $A_{\theta} = 0$, we obtain

$$A_{\theta}(x) : m_4x^4 + m_3x^3 + m_2x^2 + m_1x + m_0 = 0,$$

(19)
where
\[
\begin{align*}
  m_0 &= (A_1 + A_2)^2 + B_2^2 - (C_1 + C_3)^2, \\
  m_1 &= 4(B_1B_2 - C_2C_3 - C_1C_2), \\
  m_2 &= -2A_1^2 + 4B_1^2 + 2C_1^2 + 2A_2^2 + 2B_2^2 - 4C_2^2 - 2C_3^2, \\
  m_3 &= 4(B_1B_2 - C_2C_3 + C_1C_2), \\
  m_4 &= (A_1 - A_2)^2 + B_2^2 - (C_1 - C_3)^2.
\end{align*}
\]

The condition \( A_0 = 0 \) becomes
\[
A_0(y) : \mu_4y^4 + \mu_3y^3 + \mu_2y^2 + \mu_1y + \mu_0 = 0,
\]
where
\[
\begin{align*}
  \mu_0 &= (A_1 + C_1)^2 + C_2^2 - (A_2 + C_3)^2, \\
  \mu_1 &= 4(B_1C_2 - B_2C_3 - A_2B_2), \\
  \mu_2 &= -2A_1^2 + 4B_1^2 + 2A_2^2 + 2C_1^2 + 2C_2^2 - 4B_2^2 - 2C_3^2, \\
  \mu_3 &= 4(B_1C_2 - B_2C_3 + A_2B_2), \\
  \mu_4 &= (A_1 - C_1)^2 + C_2^2 - (A_2 - C_3)^2.
\end{align*}
\]

In both cases, we obtain quartic polynomials that depend only on the dimensional parameters of the RRSS linkage. In what follows, we examine the conditions under which these two quartics have real solutions. First, we summarize the procedure used to classify quartic polynomials in terms of their roots.

3. The roots of a quartic polynomial

A quartic polynomial \( P(x) \) with real coefficients must have complex roots that appear in complex conjugate pairs. Thus, it will have zero, two, or four real roots. We can determine which of these cases will occur by computing discriminants obtained from its resultant matrix [14].

The first step is to simplify the form of \( P(z) \) by dividing by \( a_4 \) and eliminating the cubic term using the transformation \( x = z - a_3/(4a_4) \), so
\[
P(x) : a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0,
\]
is transformed to
\[
P(z) : z^4 + uz^2 + vz + w = 0,
\]
where
\[
\begin{align*}
  u &= (8a_5a_4 - 3a_3^2)/8a_4^2, \\
  v &= (a_3^2 - 4a_2a_3a_4 + 8a_1a_4^2)/8a_4^3, \\
  w &= (-3a_4^3 + 16a_2a_3^2a_4 - 64a_1a_3a_4^2 + 256a_0a_4^3)/256a_4^4.
\end{align*}
\]
3.1. The discriminants

We now construct the resultant matrix \([R]\) for the polynomial \(P(z)\) with its derivative \(P'(z)\) [14]:

\[
[R] = \begin{bmatrix}
1 & 0 & u & v & w & 0 & 0 & 0 \\
0 & 4 & 0 & 2u & w & 0 & 0 & 0 \\
0 & 1 & 0 & u & w & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 2u & w & 0 & 0 \\
0 & 1 & 0 & u & v & w & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 2u & w & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & u & v \\
0 & 0 & 0 & 0 & 4 & 0 & 2u & w
\end{bmatrix}
\]

(26)

The \(2k \times 2k\) central minors, \(k = 1, 2, 3, 4\) shown above define the four discriminants that we use to distinguish the real roots.

Denote these discriminants as \(\Delta_k\), so we have

\[
\begin{align*}
\Delta_1 &= 4, \\
\Delta_2 &= -8u, \\
\Delta_3 &= -8u^3 - 36v^2 + 32uw, \\
\Delta_4 &= -4u^3v^2 - 27v^4 + 16u^4w + 144uv^2w - 128u^2w^2 + 256w^3.
\end{align*}
\]

(27)

Ting and Dou [11] and Williams and Reinholtz [13] construct similar discriminants. The \(q, L\) and \(\Delta\) in Williams and Reinholtz correspond to our \(u, \Delta_3/4\) and \(\Delta_4\) respectively.

3.2. The sign list

Properties of the roots of the quartic equation can be determined from the signs of \(\Delta_1, \Delta_2, \Delta_3\) and \(\Delta_4\). Yang et al. [14] use the list of the signs of these discriminants, \([s_1, s_2, s_3, s_4]\) where \(s_i = +1, -1, 0, i = 1, 2, 3, 4\), depending on whether \(\Delta_i\) is positive, negative or zero, to classify the real roots of the polynomial. A complex conjugate pair of roots is associated with each sign change in this list.

If a discriminant takes the value zero between any two non-zero values in the list \([s_1, s_2, s_3, s_4]\), then the zero is set to ‘−1’ to define a revised sign list. Yang provides the following theorem:

**Theorem.** Given a polynomial \(f(x)\) with real coefficients,

\[
f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0
\]

if the number of the sign changes (e.g. −1 to +1 or +1 to −1) of the revised sign list
has no complex roots yet only two distinct real roots. The multiplicities of these roots, \( P \) polynomial

Table 1

simultaneously, we find that are easily distinguished using the coefficient real root with multiplicity 4.

real roots, \{2,2\} means two distinct real roots, both with multiplicity 2, and \{4\} means a single real root of \( f \) equals \( l - 2v \). For a quartic the number of sign changes \( v \) can be 0, 1, or 2.

Note that \( \Delta_1 = 4 \), therefore \( s_1 \) always equals 1. Thus, the existence of real roots for the quartic polynomial \( P(z) \) can be determined from the values of \( s_2, s_3 \) and \( s_4 \).

### 3.3. Classifying the system of roots

Table 1 shows all of the revised lists that are obtained for the quartic polynomial. We use the symbols \( \land \) and \( \lor \) to denote logical “and” and “or” that ensure various signs of the discriminants.

For example, the revised sign lists for case 0a has two sign changes which means that the polynomial has two distinct pairs of complex roots, therefore there are no real roots. Note that the original sign lists for this case can be written as \([1, s_2, s_3, 1]\). Because \( s_2 \) and \( s_3 \) cannot be positive simultaneously, we find that \( \Delta_4 > 0 \) and either \( \Delta_3 \leq 0 \) or \( \Delta_2 \leq 0 \), which we write as \( (\Delta_2 \leq 0 \lor \Delta_3 \leq 0) \land \Delta_4 > 0 \).

We follow this approach to study all the cases of real roots of a quartic polynomial shown in Table 1. The cases are ordered by the number of real roots. Note, the numbers in braces indicate the multiplicity of the real roots. For instance, \{0\} means no real root, \{1,1\} means two distinct real roots, \{2,2\} means two distinct real roots, both with multiplicity 2, and \{4\} means a single real root with multiplicity 4.

We further refine the classification of four real roots in order to distinguish two cases associated with the revised sign \([1, 1, 0, 0]\). According to the Theorem, \( l = 2 \) and \( v = 0 \), therefore the quartic has no complex roots yet only two distinct real roots. The multiplicities of these roots, \( n_1 \) and \( n_2 \), must add to four, but this can be achieved in two ways, \( n_1 = n_2 = 2 \) or \( n_1 = 1, n_2 = 3 \). The cases are easily distinguished using the coefficient \( v \) in \( P(z) \), because if \( v = 0 \), then \( n_1 = n_2 = 2 \). Otherwise, for \( v \neq 0 \), we have \( n_1 = 1, n_2 = 3 \). These are denoted cases 4c and 4d, respectively.

### Table 1

**Real roots of a quartic polynomial**

<table>
<thead>
<tr>
<th>Case</th>
<th>Real roots</th>
<th>Discriminant conditions</th>
<th>Revised sign list</th>
</tr>
</thead>
<tbody>
<tr>
<td>0a</td>
<td>{0}</td>
<td>((\Delta_2 \leq 0 \lor \Delta_3 \leq 0) \land \Delta_4 &gt; 0)</td>
<td>([1, -1, -1, 1] ) or ([1, -1, 1, 1])</td>
</tr>
<tr>
<td>0b</td>
<td>{0}</td>
<td>(\Delta_2 &lt; 0 \land (\Delta_3 = \Delta_4 = 0))</td>
<td>([1, -1, 0, 0])</td>
</tr>
<tr>
<td>2a</td>
<td>{1,1}</td>
<td>(\Delta_4 &lt; 0)</td>
<td>([1, -1, -1, 1] ) or ([1, 1, 1, -1])</td>
</tr>
<tr>
<td>2b</td>
<td>{2}</td>
<td>(\Delta_3 &lt; 0 \land \Delta_4 = 0)</td>
<td>([1, -1, -1, 0])</td>
</tr>
<tr>
<td>4a</td>
<td>{1,1,1,1}</td>
<td>(\Delta_2 &gt; 0 \land \Delta_3 &gt; 0 \land \Delta_4 &gt; 0)</td>
<td>([1, 1, 1, 1])</td>
</tr>
<tr>
<td>4b</td>
<td>{1,1,2}</td>
<td>(\Delta_2 &gt; 0 \land \Delta_3 &gt; 0 \land \Delta_4 = 0)</td>
<td>([1, 1, 1, 0])</td>
</tr>
<tr>
<td>4c</td>
<td>{2,2}</td>
<td>(\Delta_2 &gt; 0 \land (\Delta_3 = \Delta_4 = 0) \land v = 0)</td>
<td>([1, 1, 0, 0])</td>
</tr>
<tr>
<td>4d</td>
<td>{1,3}</td>
<td>(\Delta_2 &gt; 0 \land (\Delta_3 = \Delta_4 = 0) \land v \neq 0)</td>
<td>([1, 1, 0, 0])</td>
</tr>
<tr>
<td>4e</td>
<td>{4}</td>
<td>(\Delta_2 = \Delta_3 = \Delta_4 = 0)</td>
<td>([1, 0, 0, 0])</td>
</tr>
</tbody>
</table>
4. The types of crank movement

We now apply the classification of the real roots to the limit polynomial for the input crank, $A_\theta$, given in Eq. (19). The coefficients of this polynomial are $m_i$ as shown in Eq. (20). Denote the four discriminants as $d_k$, $k = 1, 2, 3, 4$, respectively. Note that $d_1$ is non-zero and always positive, so we use the values $\text{sgn}(d_k)$, $k = 2, 3, 4$, and the coefficient $v$ of the transformed polynomial to classify the roots.

It is interesting to note that while the roots of the limit polynomial determine the extreme positions of the crank, the sign of the coefficient $m_4$ determines the range of movement. Since $A_\theta(x) > 0$ is required for closure of the linkage, we can use the sign of $m_4 > 0$ to determine between which set of roots $A_\theta(x)$ is positive and crank movement is feasible.

Fig. 2 shows the two cases for $A_\theta(x)$ that has four distinct real roots $\theta_i$, $i = 1, 2, 3, 4$. For $m_4 > 0$, the feasible ranges of movement are between $\theta_2$ and $\theta_3$, and between $\theta_4$ and $\theta_1$. While for $m_4 < 0$, the feasible ranges of movement are from $\theta_1$ to $\theta_2$, and from $\theta_3$ to $\theta_4$.

If no roots exists, then the sign of $m_4$ distinguishes between a fully rotatable crank, and a crank that cannot close the linkage. See Fig. 4 for the range of motion of all the cases in Table 2.

4.1. The three basic cases

There are three basic cases that describe the movement of the input crank that are distinguished by the number of real roots of the limit polynomial $A_\theta$. They are (i) the fully rotatable crank which corresponds to no real roots, (ii) the non-Grashof rocker for the case of two real roots, and (iii) the Grashof rocker which is given by four real roots.

4.1.1. Crank: case 0a

The condition for the input crank to be fully rotatable is

\[(d_2 \leq 0 \lor d_3 \leq 0) \cap d_4 > 0.\]  

In this case, Eq. (19) has no real roots. Notice that in order for the linkage to close we must have $m_4 > 0$. If $m_4 < 0$, then linkage does not close. We denote this case as case 0a’ in Table 2 and Fig. 4.

![Fig. 2. Plots of $A_\theta$ versus $x$.](image-url)
The link dimensions of an RRSS linkage with this kind of input crank are

\[ a = 0.5, \quad b = 2, \quad h = 1, \quad p = -3.72972, \quad q = -3.8115, \quad r = 0.5, \quad \alpha = 30^\circ \]

Projections of the trajectory of the joint \( S_1 \) in the \( X'Z' \) and \( X'Z'' \) planes of the coupler frame \( M \) are shown in Fig. 3. This trajectory is symmetric about the plane \( \mathcal{P} \) in the coupler; recall that this is the \( X'Z' \) coordinate plane in \( M \).

4.1.2. Non-Grashof rocker: case 2a

The condition for the input crank to be a non-Grashof rocker is

\[ d_4 < 0. \quad (29) \]

An RRSS linkage with this type of input crank has the dimensions

\[ a = 1.48148, \quad b = 0.740741, \quad h = 1, \quad p = 0.185185, \quad q = 0.37037, \quad r = 0.462963, \quad \alpha = 60^\circ. \]

The input link rocks between the two limits \( \theta_1 = 123.782^\circ \) and \( \theta_2 = -83.0457^\circ \). See Fig. 4. The sign of \( m_4 \) determines the side of the circle that is feasible for movement between these angles. See Fig. 3 for the plot of the trajectory of \( S_1 \) in \( M \).

4.1.3. Grashof rocker: case 4a

The condition for the input crank to be a Grashof rocker is

\[ d_2 > 0 \land d_3 > 0 \land d_4 > 0. \quad (30) \]
Fig. 3. Trajectory of \( S_1 \) in the moving frame \( M \) for all cases.
An example RRSS linkage with this input crank is

\[ a = 0.502863, \quad b = 2.84947, \quad h = 1, \quad p = 3.94542, \quad q = 2.88878, \quad r = 1.28271, \quad x = 42.93^\circ. \]

The four limits are \( \theta_1 = -135^\circ, \theta_2 = -60^\circ, \theta_3 = 45^\circ, \) and \( \theta_4 = 120^\circ. \) The two distinct movement ranges are determined by evaluating the sign of \( m_4 \) between these limits (Fig. 4). See Fig. 3 for the trajectory of \( S_1 \) in \( M. \) The example shows us that we are able to design an RRSS linkage with specified limits.
4.2. The boundary cases

The following six cases are defined by the condition $d_4 = 0$ which we consider to be the boundary between Grashof and non-Grashof movement. We introduce the special crank which is essentially identical to the rotatable crank above, the boundary Grashof rocker and the boundary non-Grashof rocker both of which rock between two limit angles, and three cases of boundary cranks that fully rotate, but have one or two singular angles.

4.2.1. Special crank: case 0b

There is a rotatable crank that has a complex root of multiplicity two rather than two separate complex roots. This case is defined by the condition

$$d_2 < 0 \land d_3 = d_4 = 0.$$  \hfill (31)

If $m_4 > 0$ then the linkage can be assembled and the crank fully rotates. If $m_4 < 0$ (case 0b'), the linkage does not close. See Fig. 4.

An example of this kind of input crank is

$$a = -0.636678, \quad b = 1.20703, \quad h = 1, \quad p = 0.0831286, \quad q = 0.332818, \quad r = 0.1, \quad \alpha = -72.0191^\circ.$$  

See Fig. 3 for the trajectory of $S_1$ in the $M$.

4.2.2. Boundary Grashof rocker: case 4b

The Grashof rocker generally has two ranges of movement between pairs of four limit points. A boundary Grashof rocker occurs when two of these limit points merge to form double point. The condition for this case is

$$d_2 > 0 \land d_3 > 0 \land d_4 = 0.$$  \hfill (32)

In this case, the crank rocks between one pair of limit points, and mayor may not pass through the double limit point depending on the sign of $m_4$. If the range contains this double limit, then the linkage does not close at this point. On the other hand, if the range does not contain it, then the linkage can be closed only at this point. See Fig. 4.

An example of the kind of input crank is given by

$$a = 0.201236, \quad b = 0.594043, \quad h = 1, \quad p = -0.209952, \quad q = 0.928983, \quad r = 0.835514, \quad \alpha = -76.8927^\circ.$$  

The four limits are $\theta_1 = -43.2814^\circ, \theta_2 = \theta_3 = 1.28335^\circ, \theta_4 = 46.5681^\circ$. See Fig. 3 for the plot of the trajectory of $S_1$ in the $X'Z'$ and $X'Y'$ planes.

4.2.3. Boundary non-Grashof rocker: case 4d

This case arises when the limit polynomial has a triple limit point. The condition for this case is

$$d_2 > 0 \land d_3 = d_4 = 0 \land v \neq 0.$$  \hfill (33)

In this case the crank rocks between the single limit point and the triple limit point (Fig. 4).
An example of the kind of input crank is
\[ a = 0.348024, \quad b = 0.857077, \quad h = 1, \quad p = 0.19262, \quad q = 2.95437, \quad r = 2.16104, \]
\[ \alpha = 75.7622^\circ. \]

The limits are \( \theta_1 = \theta_2 = \theta_3 = 74.4609^\circ \) and \( \theta_4 = 111.006^\circ \). See Fig. 3 for the plot of the trajectory of \( S_1 \) in the \( X'Z' \) and \( X'Y' \) planes.

### 4.2.4. Boundary crank: case 2b

The following three cases are called boundary cranks because they almost fully rotate, with the exception of one or two angles.

For the case defined by the conditions
\[ d_3 < 0 \land d_4 = 0, \]
the input crank has a double limit. Thus, if \( m_4 > 0 \) the linkage reaches every angle, except the value associated with the double limit. On the other hand, if \( m_4 < 0 \) (case 2b'), the linkage can only be assembled at the limit. An RRSS linkage with type of input crank has the dimensions
\[ a = 0.218017, \quad b = 0.420805, \quad h = 1, \quad p = 0.99674, \quad q = 0.072946, \quad r = 0.20981, \]
\[ \alpha = 9.00002^\circ. \]

The double limit is \( \theta_1 = \theta_2 = 39.9008^\circ \). See Fig. 3 for the plot of the trajectory of \( S_1 \) in the \( X'Z' \) and \( X'Y' \) planes.

### 4.2.5. Boundary crank: case 4c

Another example of a boundary crank has two double limits. The condition for this case is
\[ d_2 > 0 \land d_3 = d_4 = 0 \land v = 0. \]

If \( m_4 > 0 \), then the input crank can reach all angles except the two double limits. If \( m_4 < 0 \) (case 4c') then it can be assembled only at these two limits (Fig. 4).

An example of this type of input crank is seen in the RRSS linkage with dimensions
\[ a = 0.5, \quad b = 1, \quad h = 1, \quad p = 0.866025, \quad q = 1.5, \quad r = 0.5, \quad \alpha = 30^\circ. \]

The four limits are \( \theta_1 = \theta_2 = -90^\circ \) and \( \theta_3 = \theta_4 = 36.8699^\circ \). See Fig. 3 for the trajectory of \( S_1 \) in the \( X'Z' \) and \( X'Y' \) planes.

### 4.2.6. Boundary crank: case 4e

In this case, there is a single limit with multiplicity 4. The condition is
\[ d_2 = d_3 = d_4 = 0 \]

As we have seen above, if \( m_4 > 0 \) then the input crank can reach every angle except at the limit point. However, If \( m_4 < 0 \) (case 4e'), the linkage can only be assembled at the limit. See Fig. 4.

An example of the kind of input crank is given by
\[ a = -0.757983, \quad b = 1.61837, \quad h = 1, \quad p = -0.668518, \quad q = -0.656526, \]
\[ r = -0.42564, \quad \alpha = 65.3378^\circ. \]
The limiting angle is \( \theta_i = -12.4725^\circ \), \( i = 1, 2, 3, 4 \). See Fig. 3 for the plot of the trajectory of \( S_i \) in the \( X'Z' \) and \( X'Y' \) planes.

4.3. Range of movement of the coupler angle

The limits to the movement of the coupler angle are the roots of the limit polynomial \( \Delta_\phi \) (21). In this case, we denote the four discriminants of the limit polynomial as \( \delta_k \), respectively.

For example, we can identify three basic types of movement available to the coupler link of an RRSS linkage.

(1) The condition for the coupler to be a fully rotatable relative to the input crank is

\[
(\delta_2 \leq 0 \lor \delta_3 \leq 0) \land \delta_4 > 0 \land \mu_4 > 0.
\]

(37)

(2) The condition for the coupler to be a non-Grashof rocker is

\[
\delta_4 < 0.
\]

(38)

(3) The condition for the coupler to be a Grashof rocker is

\[
\delta_2 > 0 \land \delta_3 > 0 \land \delta_4 > 0.
\]

(39)

The remaining cases are obtained from an analysis identical to what is provide above for the input crank.

5. Classification of RRSS linkages

RRSS linkages can be classified by identifying the type of movement of the input crank and coupler around the two R-joints. Considering only the three basic crank types, it would appear that nine combinations of RRSS linkages are possible. However, we show here that only five of these combination can exist, because the signs of the discriminants \( d_4 \) and \( \delta_4 \) for the two limit polynomials are always the same.

5.1. Relationship between \( d_4 \) and \( \delta_4 \)

The expression for the discriminant \( \Delta_4 \) in (27) for a quartic polynomial can be expanded in terms of the coefficients \( a_i \) to obtain

\[
\Delta_4 = \Lambda / a_4^6,
\]

(40)

where

\[
\Lambda = a_3^2(a_1^2(a_1^2 - 4a_0a_2) - 2a_1(2a_1^2 - 9a_0a_2)a_3 - 27a_0^2a_3^2)
\]

\[
+ 2(-2a_1^2(a_1^2 - 4a_0a_2) + a_1a_2(9a_1^2 - 40a_0a_2)a_3 - 3a_0(a_1^2 - 24a_0a_2)a_3^2)a_4
\]

\[
- (27a_4^4 - 144a_0a_1^2a_2 + 128a_0^3a_2^2 + 192a_0^2a_1a_3)a_4^2 + 256a_0^3a_4^3.
\]

(41)
The algebraic manipulation software Mathematica allows us to evaluate this expression using the coefficients \( m_i \) for the limit polynomial \( A_i(x) = 0 \), and the coefficients \( \mu_i \) for the polynomial \( A_\phi(y) = 0 \). Comparing the results, we find that they are exactly the same, that is

\[
\Lambda = d_4 m_4^6 = \delta_4 \mu_4^6.
\]

This means that the signs of \( d_4 \) and \( \delta_4 \) must be the same.

This result shows that the input and the coupler links must both be either Grashof or non-Grashof. We cannot have a Grashof input crank and a non-Grashof coupler movement, or any similar mixed case. In what follows, we use the fact that \( \text{sgn}(\Lambda) = \text{sgn}(d_4) = \text{sgn}(\delta_4) \).

5.2. RRSS linkage types

The criterion for a Grashof RRSS linkage is \( \Lambda > 0 \). Without considering the boundary linkages, we can distinguish the following five linkage types. Four are combinations of Grashof cranks and rockers, and the last is constructed from two non-Grashof rockers. A summary is shown in the Table 3.

### 5.2.1. The double crank

In this case, both the input and coupler links of the RRSS linkage can fully rotate. An example of this linkage has the dimensions

\[
a = 0.5, \quad b = 1, \quad h = 1, \quad p = 0, \quad q = 0.4, \quad r = -0.7, \quad \alpha = 45^\circ.
\]

The plot of the coupler angle \( \phi \) as a function of \( \theta \) is shown in Fig. 5.

### 5.2.2. The crank–rocker

This RRSS linkage has an input crank that can fully rotate and a coupler crank that is a Grashof rocker. An example of this linkage has the dimensions

\[
a = 1.5, \quad b = 1, \quad h = 1, \quad p = 0.2, \quad q = 0.4, \quad r = 0.3, \quad \alpha = 45^\circ.
\]

The plot of \( \phi \) as a function of \( \theta \) is shown in Fig. 5.

### 5.2.3. The rocker–crank

In this case the coupler link is fully rotatable while the input link is a Grashof rocker. An example of this kind of linkage has the dimensions

### Table 3

<table>
<thead>
<tr>
<th>Classification of RRSS linkages</th>
<th>Condition</th>
<th>Sub-types</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grashof linkage</td>
<td>( \Lambda &gt; 0 )</td>
<td>Double crank</td>
<td>( m_4 &gt; 0 \land (d_2 \leq 0 \lor d_3 \leq 0) \land \mu_4 &gt; 0 \land (\delta_2 \leq 0 \lor \delta_3 \leq 0) )</td>
</tr>
<tr>
<td>Crank–rocker</td>
<td>( m_4 &gt; 0 \land (d_2 \leq 0 \lor d_3 \leq 0) \land (\delta_2 &gt; 0 \land \delta_3 &gt; 0) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rocker–crank</td>
<td>( (d_3 &gt; 0 \land d_1 &gt; 0) \land \mu_4 &gt; 0 \land (\delta_2 \leq 0 \lor \delta_3 \leq 0) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Double rocker</td>
<td>( (d_2 &gt; 0 \land d_3 &gt; 0) \land (\delta_2 &gt; 0 \land \delta_3 &gt; 0) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Non-Grashof linkage</td>
<td>( \Lambda &lt; 0 )</td>
<td>Double rocker</td>
<td></td>
</tr>
<tr>
<td>Boundary linkage</td>
<td>( \Lambda = 0 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This result shows that the input and the coupler links must both be either Grashof or non-Grashof. We cannot have a Grashof input crank and a non-Grashof coupler movement, or any similar mixed case. In what follows, we use the fact that \( \text{sgn}(\Lambda) = \text{sgn}(d_4) = \text{sgn}(\delta_4) \).
The \( h / h \) plot is shown in Fig. 5.

5.2.4. The Grashof double rocker

The final Grashof case occurs when both the input and coupler cranks are Grashof rockers. An example of this kind of linkage is

\[
a = 1, \quad b = 0.5, \quad h = 1, \quad p = 0.5, \quad q = 0.2, \quad r = 1, \quad \alpha = 30^\circ.
\]

The \( \theta - \phi \) plot is shown in Fig. 5.

5.2.4. The Grashof double rocker

The final Grashof case occurs when both the input and coupler cranks are Grashof rockers. An example of this kind of linkage is

\[
a = 1.5, \quad b = 1, \quad h = 1, \quad p = 0.6, \quad q = 0.4, \quad r = 1.7, \quad \alpha = 45^\circ.
\]

The plot of \( \phi \) versus \( \theta \) is shown in Fig. 5.
5.2.5. Non-Grashof double rocker

The final linkage type that we present is the non-Grashof double rocker, in which both the input and coupler links are non-Grashof rockers. An example of this kind of linkage has the dimensions

\[ a = 1.5, \quad b = 1, \quad h = 1, \quad p = 0.5, \quad q = 0.4, \quad r = 1, \quad \alpha = 45^\circ. \]

The \( \theta-\phi \) plot of the example is shown in Fig. 5.

5.3. The planar 4R linkage

The dimensions of the RRSS linkage can be selected so that it has the same movement as a planar 4R linkage. This is done by setting \( \alpha, p \) and \( q \) to zero. In this case, the discriminants for the limit polynomial of the input crank take the form:

\[
\begin{align*}
    d_2 &= (T_3 T_4 T_7 T_8 T_9)(-8/m_4^2), \\
    d_3 &= (T_3 T_4 T_7 T_8 T_9)(-2048a^2b^2h^2r^2/m_4^4), \\
    d_4 &= (T_1 T_2 T_3 T_4 T_5 T_6 T_7 T_8)(1048576a^4b^4h^4r^4/m_4^6),
\end{align*}
\]  

where,

\[
\begin{align*}
    T_1 &= -a + b - h + r, \\
    T_2 &= -a - b + h + r, \\
    T_3 &= -a + b + h - r, \\
    T_4 &= a + b + h + r, \\
    T_5 &= -a + b + h + r, \\
    T_6 &= a + b + h - r, \\
    T_7 &= a - b + h + r, \\
    T_8 &= a + b - h + r, \\
    T_9 &= T_1 T_2 T_3 T_4 + T_5 T_6 T_7 T_8.
\end{align*}
\]

Furthermore, we have

\[
m_4 = T_3 T_4 T_7 T_8.
\]

The parameters \( T_1, T_2, T_3 \) and \( T_4 \) are identical to those used by McCarhty [4] to classify planar 4R linkages.

We can assume that the dimensions \( a, b, h \) and \( r \) are positive lengths, in which case the parameters \( T_4, T_5, T_6, T_7, T_8 \) are always positive. Otherwise, the linkage will not close. Therefore, we can classify planar 4R linkages using three parameters \( T_1, T_2 \) and \( T_3 \). The result is the same classification provided in [4]. Thus, the classification of planar 4R linkages is obtained as a special case of our classification of the RRSS linkage.
6. A type map for RRSS linkages

The conditions defining the RRSS linkage types can be used to map the type of linkage that results from various combinations of dimensional parameters. This type map is useful in the dimensional synthesis of linkages to guide the designer’s selection of candidate designs.

It is interesting to examine the distribution of the input crank type as a function of the parameters $r$ and $p$ that locate the fixed joint $S_1$. We consider the linkage with dimensions

![Type Map for RRSS Linkages](image)

Fig. 6. A map of the input crank types as a function of $r$ and $p$.

![Example of Type Map](image)

Fig. 7. An example of a type map for RRSS linkages designs.
\[ a = 1, \quad b = 1/3, \quad q = 2/5, \quad \alpha = \pi/4, \quad \text{and} \quad h = 1. \]

The result is the discriminants \( d_2, d_3, d_4 \) and \( m_4 \) are functions of \( r \) and \( p \). Their contours can be plotted in the \( XZ \) plane of \( F \). See Fig. 6.

The conditions for both the input and coupler link types can be combined to obtain a complete map for the RRSS linkage types. For the example, using the linkage dimensions

\[ a = 1/4, \quad b = 1, \quad q = 2/5, \quad \alpha = \pi/4, \quad \text{and} \quad h = 1 \]

and varying \( r \) and \( p \), we can obtain the type map of the RRSS linkages. See Fig. 7. In this example, only four of the five basic types appear. They are the double crank, the crank–rocker, the rocker–crank and the non-Grashof double rocker.

Using the same dimensions but setting the parameter \( h = 1/4 \), we obtain a type map consisting only of Grashof and non-Grashof double rockers. See Fig. 8.

7. Conclusions

This paper presents a classification of RRSS linkages based on the roots of limit polynomials associated with the rotation of the input and coupler links. The real roots of these polynomials determine the existence of limiting angles that define the range of movement of these links. The classification of the various combinations of these ranges of movement of both links yields a classification of RRSS linkages. We obtain five basic cases, and provide examples of maps of the linkage type as a function of dimensional parameters. This type map is useful in the computer-aided design of RRSS mechanisms.

Acknowledgements

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References