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Dimensional Synthesis of Bennett Linkages

Three position synthesis of a spatial RR chain yields two solutions that can be assembled to form a Bennett linkage. In this paper, we show that the cylindroid associated with Bennett's linkage can be generated directly from the three specified positions, and used to simplify the design equations. Analysis of these equations shows that there are always two solutions and that their symmetry yields a Bennett linkage. An example design is presented. [DOI: 10.1115/1.1539507]

1 Introduction

Three position synthesis of a spatial RR chain has been shown to yield two solutions that combine to form a spatial 4R linkage, known as Bennett's linkage. Initially, Veldkamp [1] obtained this result for three instantaneous positions. Suh [2] obtained numerical results that showed that the solution of the finite position synthesis problem also yielded two solutions that formed a Bennett linkage. Finally, Tsai and Roth [3] reduced ten quadratic design equations to a single polynomial and showed that it always has two solutions.

Recent study of Bennett's linkage has focused on the set of finite displacement screws that define the movement of the coupler [4]. The axes of these screws form a ruled surface known as a cylindroid. We show that this cylindroid defines a coordinate frame that simplifies the design equations for spatial RR synthesis. The result is three linear equations and one cubic polynomial in four design parameters.

2 The Bennett Linkage

Bennett's linkage [5] moves with one degree of freedom, due to its special geometry. The mobility of a general 4R spatial linkage is obtained by applying Gruebler's criterion,

$$M=6(n-1)-\sum_{k=1}^{m} p_k c_k=6(4-1)-4\cdot 5=-2. \tag{1}$$

Thus, in general, this assembly of links and joints forms a structure.

The twist angles and link lengths of the opposite sides of Bennett's linkage, (α,a) and (γ,g), must be equal, see Fig. 1. This together with the condition

$$\frac{\sin\alpha}{a}=\frac{\sin\gamma}{g}, \tag{2}$$

ensures that the linkage moves with one degree of freedom.

We use the design equations for the spatial RR chain in order to design a Bennett linkage. The spatial RR chain consists of a fixed revolute axis $\mathsf{G}=(\mathbf{G},\mathbf{B}\times\mathbf{G})^T=(\mathbf{G},\mathbf{R})^T$ connected to a moving revolute axis $\mathsf{W}^i=(\mathbf{W}^i,\mathbf{P}^i\times\mathbf{W}^i)^T=(\mathbf{W}^i,\mathbf{V}^i)^T$ by a rigid link. The vector $\mathbf{G}$ denotes the direction of the fixed axis and $\mathbf{B}$ is a point on this line. The vector $\mathbf{W}^i$ is the direction of the moving axis in the ith position and $\mathbf{P}^i$ a point on this line. We choose $\mathbf{B}$ and $\mathbf{P}^i$ to be the intersection points of these lines with the common normal N to the two axes, Fig. 2.

The RR chain is defined by the directions $\mathbf{G}$ and $\mathbf{W}^1$ and the points $\mathbf{B}$ and $\mathbf{P}^1$ that locate the lines G and W^1 in space. This yields a total of ten design parameters.

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3 Geometry of the RR Chain

The workspace of the RR chain is the set of displacements $[D(\theta,\phi)]$ defined by the kinematics equations,

$$[D]=[G][Z(\theta,0)][X(\alpha,a)][Z(\phi,0)][H], \tag{3}$$

where $[G]$ and $[H]$ are the initial and final transformations from the fixed frame to the fixed axis and from the moving axis to the end effector, respectively. This workspace plays an important role in simplifying the design equations.

Choosing a reference configuration $[D_1]$, we can construct the set of the relative displacements $[D_{1i}]=[D_i][D_1]^{-1}$,

$$[D_{1i}]=[T(\Delta\theta_i,\mathsf{G})][T(\Delta\phi_i,\mathsf{W}^1)], \tag{4}$$

where

$$\begin{aligned}[T(\Delta\theta,\mathsf{G})]&=[G][Z(\theta,0)][Z(\theta_0,0)]^{-1}[G]^{-1},\\ [T(\Delta\phi,\mathsf{W}^1)]&=([G][[Z(\theta_0,0)][X(\alpha,a)])[Z(\phi,0)]\\ &\quad [Z(\phi_0,0)]^{-1}([G][[Z(\theta_0,0)][X(\alpha,a)])^{-1}.\end{aligned} \tag{5}$$

This equation defines the workspace of RR chain directly in terms of the the rotations $\Delta\theta$ and $\Delta\phi$ about the fixed and moving axes, G and W^1, respectively.

Introduce the dual quaternions (Bottema and Roth [6]) that represent the relative displacements $[T(\Delta\theta,\mathsf{G})]$ and $[T(\Delta\phi,\mathsf{W}^1)]$ given by

$$\hat{G}\left(\frac{\Delta\theta}{2}\right)=\cos\left(\frac{\Delta\theta}{2}\right)+\sin\left(\frac{\Delta\theta}{2}\right)\mathsf{G},$$

and

$$\hat{W}^1\left(\frac{\Delta\phi}{2}\right)=\cos\left(\frac{\Delta\phi}{2}\right)+\sin\left(\frac{\Delta\phi}{2}\right)\mathsf{W}^1. \tag{6}$$

The displacement $\hat{S}_{1i}$ of the end-link is computed using the dual quaternion product $\hat{S}_{1i}=\hat{G}\hat{W}^1$ [7], which yields the dual scalar

$$\cos\left(\frac{\hat{\psi}_{1i}}{2}\right)=\cos\frac{\Delta\theta}{2}\cos\frac{\Delta\phi}{2}-\sin\frac{\Delta\theta}{2}\sin\frac{\Delta\phi}{2}\mathsf{G}\cdot\mathsf{W}^1, \tag{7}$$

and the dual vector

$$\begin{aligned}\sin\left(\frac{\hat{\psi}_{1i}}{2}\right)\mathsf{S}_{1i}&=\sin\frac{\Delta\theta}{2}\cos\frac{\Delta\phi}{2}\mathsf{G}+\sin\frac{\Delta\phi}{2}\cos\frac{\Delta\theta}{2}\mathsf{W}^1\\ &\quad+\sin\frac{\Delta\theta}{2}\sin\frac{\Delta\phi}{2}\mathsf{G}\times\mathsf{W}^1.\end{aligned} \tag{8}$$

The dual angle $\hat{\psi}_{1i}$ defines the relative rotation ψ_{1i} and translation t_{1i} of the end-link along the screw axis S_{1i}. Equation (8) yields the screw axes of displacements reachable by the RR chain. This is called a third order screw system by Huang [8].

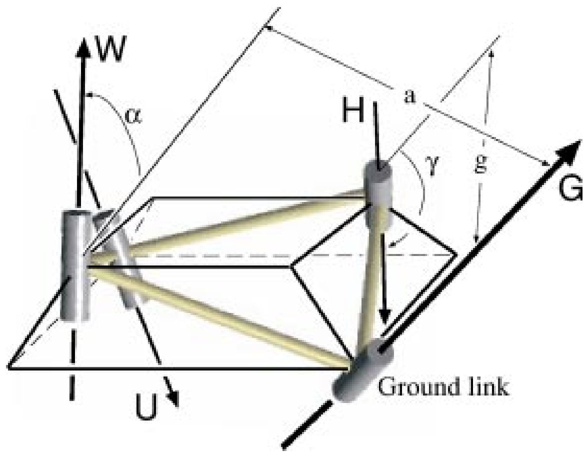

Fig. 1 A Bennett linkage

If the RR chain is part of a Bennett linkage, then Hunt [9] shows that the values of θ and ϕ related by

$$\tan\frac{\phi}{2}=\frac{\sin\frac{\alpha+\gamma}{2}}{\sin\frac{\alpha-\gamma}{2}}\tan\frac{\theta}{2}=K\tan\frac{\theta}{2}, \tag{9}$$

where K is the constant obtained from the dimensions α and γ. We translate this into a condition on the relative angles $\Delta\theta=\theta_i-\theta_1$ to obtain

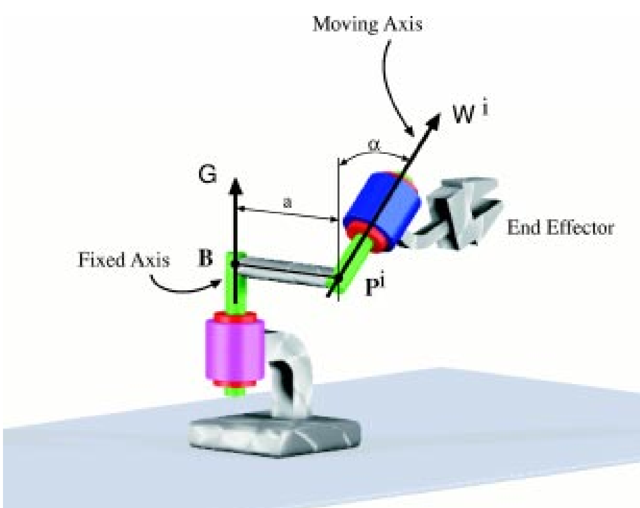

Fig. 2 A spatial RR robot

$$\tan\frac{\Delta\phi}{2}=\frac{K\tan\frac{\Delta\theta}{2}\left(1+\tan^2\frac{\theta_1}{2}\right)}{1+(K^2-1)\tan\frac{\theta_1}{2}\tan\frac{\Delta\theta}{2}+K^2\left(\tan^2\frac{\theta_1}{2}\right)}. \tag{10}$$

Substitute this relation into the equation defining the screw axes of the RR chain, Eq. (8), in order to obtain the set of screw axes for Bennett's linkage,

$$\frac{a\sin\alpha\sin\frac{\psi_{1i}}{2}}{a\sin\alpha\cos\frac{\psi_{1i}}{2}-\frac{t_{1i}}{2}\cos\alpha\sin\frac{\psi_{1i}}{2}}\left(1,\frac{\frac{t_{1i}}{2}}{\tan\frac{\psi_{1i}}{2}}\right)\mathsf{S}_{1i}$$

$$=\tan\frac{\Delta\theta}{2}\left(\mathsf{G}+K_\theta\mathsf{W}^1+\tan\frac{\Delta\theta}{2}K_\theta\mathsf{G}\times\mathsf{W}^1\right), \tag{11}$$

where

$$K_\theta=\frac{K(1+K_1^2)}{1+K^2K_1^2+\tan\frac{\Delta\theta}{2}K_1(K^2-1)},$$

and

$$K_1=\tan\frac{\theta_1}{2}. \tag{12}$$

Vary $\Delta\theta=\theta_i-\theta_1$ in order to obtain the set of relative screw axes that define the movement of the coupler of a Bennett linkage. It is remarkable that the set of axes of these screws forms a *cylindroid*.

4 The Cylindroid

A cylindroid is a ruled surface that has a nodal line cutting all generators at right angles. See Fig. 3. The cylindroid appears as the set of axes of a real linear combination of two screws (Hunt [9]).

If we designed the RR chain to reach the three spatial positions M_1, M_2 and M_3, then the relative screw axes S_{12} and S_{13} must lie on the cylindroid defined by Eq. (11). In fact the real linear combination of these two screws must generate this cylindroid. This is the key to our new formulation of the RR design problem.

To describe the geometry of the cylindroid, let us consider the screws obtained from the design positions,

$$\mathsf{V}_a=\sin\frac{\hat{\psi}_{12}}{2}\mathsf{S}_{12},\quad \mathsf{V}_b=\sin\frac{\hat{\psi}_{13}}{2}\mathsf{S}_{13}. \tag{13}$$

The dual number $\sin\hat{\psi}_{1i}/2=(\sin\psi_{1i}/2, t_{1i}/2\cos\psi_{1i}/2)$ defines the magnitude and pitch of these screws. From this we obtain $M_a=\sin\psi_{12}/2$ and $M_b=\sin\psi_{13}/2$ as their respective magnitudes. Their pitches are computed to be

Fig. 3 The cylindroid as viewed first from the top along the central axis, then from the side and finally, from an angle

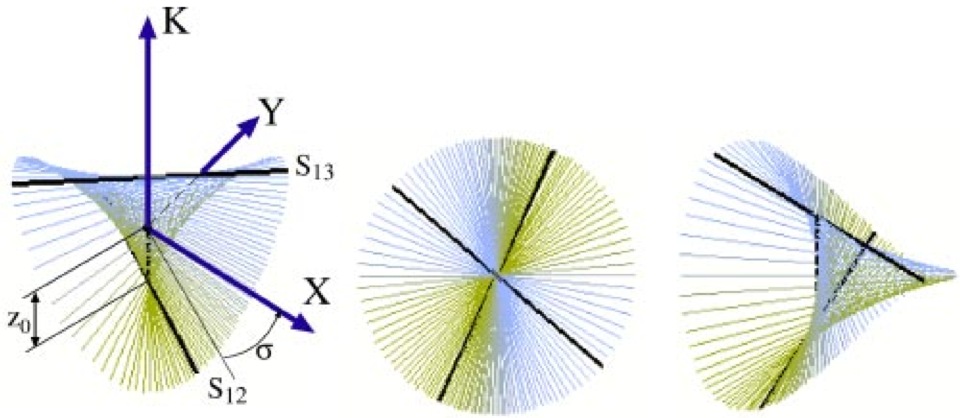

Fig. 4 The principal axes as located from the initial screws S_{12} and S_{13}

$$P_a = \frac{t_{12}}{2\tan\frac{\psi_{12}}{2}}, \quad P_b = \frac{t_{13}}{2\tan\frac{\psi_{13}}{2}}. \tag{14}$$

For reference see Parkin [10].

We now generate the cylindroid as a linear combination of the two screws V_a and V_b. For the following calculations $\hat{\delta}=(\delta,d)$ be the dual angle between S_{12} and S_{13} which are the axes of V_a and V_b.

4.1 The Principal Axes. The cylindroid has a set of *principal axes* consisting of the nodal line and a pair of rulings that intersect in a right angle. This occurs at the midpoint of the cylindroid along the nodal line [9,11]. The position of the principal axes relative to the screw V_a is defined by the dual angle $\hat{\sigma}=(\sigma,z_0)$. See Fig. 4.

The angle σ is given by

$$\tan 2\sigma = \frac{-(P_b-P_a)\cot\delta + d}{(P_b-P_a)+d\cot\delta}. \tag{15}$$

This yields two angles separated by $\pi/2$. This defines the directions of both principal axes X and Y. The offset z_0 measured from V_a to the principal axes is given by

$$z_0 = \frac{1}{2}\left(d-(P_b-P_a)\frac{\cos\delta}{\sin\delta}\right). \tag{16}$$

The principal axes of this cylindroid provide a convenient coordinate frame for our synthesis of RR chains.

5 The Design Equations

The design equations for the RR chain are obtained from the geometric constraints imposed by the link connecting the moving and fixed axes, [3,12], and [13]. In particular, the twist angle α and the distance a between these axes must remain constant during the movement, and the common normal line to both axes is constrained to pass though the same points of both axes.

5.1 The matrix formulation. Let the three design positions be defined by the 4×4 homogeneous transforms

$$[T_i] = \begin{bmatrix} & [A_i] & & \mathbf{d}_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad i=1,2,3, \tag{17}$$

where $[A_i]$ is a 3×3 rotation matrix and $\mathbf{d}_i$ is a 3×1 translation vector. We define the relative displacements $[T_{1i}]=[T_i][T_1^{-1}]$, $i=2,3$. Associated with each of these matrices is a 6×6 coordinate transformation for screws,

$$[\hat{T}_{1i}] = \begin{bmatrix} A_{1i} & 0 \\ D_{1i}A_{1i} & A_{1i} \end{bmatrix}, \tag{18}$$

where D_{1i} is the 3×3 skew-symmetric matrix obtained such that $[D_{1i}]\mathbf{y}=\mathbf{d}_{1i}\times\mathbf{y}$.

Dual vector algebra [13] allows us to represent the constant twist angle and distance constraints for the RR chain as the equations,

$$\mathsf{G}\cdot[\hat{T}_{1i}-I]\mathsf{W}^1=0, \quad i=2,3. \tag{19}$$

The real part of these equations yield two scalar direction constraints,

$$\mathbf{G}\cdot[A_{1i}-I]\mathbf{W}^1=0, \quad i=2,3, \tag{20}$$

and the dual part provides two scalar moment constraint equations

$$\mathbf{G}\cdot[A_{1i}-I]\mathbf{V}^1+\mathbf{W}^1\cdot[A_{1i}-I]^T\mathbf{R}+\mathbf{G}\cdot[D_{1i}A_{1i}]\mathbf{W}^1=0, \quad i=2,3. \tag{21}$$

These equations ensure that the dual angle $\hat{\alpha}=(\alpha,a)$ is constant in the three positions.

In order to ensure that the common normal passes through the same points $\mathbf{B}$ and $\mathbf{P}^i=[T_{1i}]\mathbf{P}^1$ on both axes, we require $\mathbf{P}^i-\mathbf{B}$ to be perpendicular to $\mathbf{G}$ and $[T_{1i}]^{-1}\mathbf{B}-\mathbf{P}^1$ to be perpendicular to $\mathbf{W}^1$ in each position. That is, we have the six equations,

$$\mathbf{G}\cdot([T_{1i}]\mathbf{P}^1-\mathbf{B})=0, \quad \mathbf{W}^1\cdot(\mathbf{P}^1-[T_{1i}]^{-1}\mathbf{B})=0, \quad i=1,2,3. \tag{22}$$

The ten equations (20), (21), (22) are solved to determine G and W^1.

5.2 The Equivalent Screw Triangle. The equivalent screw triangle formulation [14] provides another approach to the constraint equations for the RR chain. For each relative displacement, we construct the screw axis S_{1i} and a rotation angle ψ_{1i} and a translation t_{1i}. The equivalent screw triangle defines the relationship between the screw axis S_{1i} and the fixed and moving axes G and W^1.

Let $\mathbf{C}_{1i}$ be a reference point on the screw axis $\mathbf{S}_{1i}$. The direction constraint in Eq. (20) can be reformulated to yield,

$$[A_{1i}-I] = \frac{2}{1+\tan^2(\psi_{1i}/2)}[B_{1i}-B_{1i}^T B_{1i}], \tag{23}$$

where $[B_{1i}]$ is the skew-symmetric matrix corresponding to Rodrigues' vector, $\mathbf{B}_{1i}=\tan(\psi_{1i}/2)\mathbf{S}_{1i}$, such that $[B_{1i}]\mathbf{v}=\mathbf{B}_{1i}\times\mathbf{v}$. This allows us to write the direction equation as

$$k\mathbf{G}\cdot([S_{1i}]-k'[S_{1i}^T S_{1i}])\mathbf{W}^1$$
$$=k\mathbf{G}\cdot(\mathbf{S}_{1i}\times\mathbf{W}^1)-kk'((\mathbf{S}_{1i}\times\mathbf{G})\cdot(\mathbf{S}_{1i}\times\mathbf{W}^1)) \quad (24)$$

which yields the equation used by Tsai and Roth [3],

$$\tan\frac{\psi_{1i}}{2}=\frac{\mathbf{G}\cdot(\mathbf{S}_{1i}\times\mathbf{W}^1)}{(\mathbf{S}_{1i}\times\mathbf{G})\cdot(\mathbf{S}_{1i}\times\mathbf{W}^1)}. \quad (25)$$

The properties of the screw triangle also yield an alternative set of equations for the moment constraints. Notice that the geometry of the dyad triangle requires the common normal lines to G and W^i to be separated by a distance $t_{1i}/2$ along S_{1i}. Thus the component of $\mathbf{B}-\mathbf{P}^1$ in the direction S_{1i} is given by

$$(\mathbf{B}-\mathbf{P}^1)\cdot\mathbf{S}_{1i}-\frac{t_{1i}}{2}=0, \quad i=2,3. \quad (26)$$

To complete the set of design equations, we transform the expressions in Eq. (22) to obtain

$$\begin{aligned}\mathbf{G}\cdot(\mathbf{P}^1-\mathbf{B})&=0,\\ \mathbf{W}^1\cdot(\mathbf{P}^1-\mathbf{B})&=0,\end{aligned} \quad (27)$$

and

$$\begin{aligned}\mathbf{G}\cdot[I-\mathbf{S}_{1i}\mathbf{S}_{1i}^T](\mathbf{B}-\mathbf{C}_{1i})&=0,\\ \mathbf{W}^1\cdot[I-\mathbf{S}_{1i}\mathbf{S}_{1i}^T](\mathbf{P}^1-\mathbf{C}_{1i})&=0, \quad i=2,3,\end{aligned} \quad (28)$$

where $\mathbf{C}_{1i}$ is any point on $\mathbf{S}_{1i}$.

This is a set of 10 quadratic equations in ten unknowns that define the coordinates of the line G and the line W^1 in its first position.

6 Bennett Linkage Coordinates

Yu [15] introduced a coordinate frame that simplified his analysis of Bennett's linkage, which we adapt for our purposes. To identify this frame, we begin by noting that the sides of Bennett's linkage form a tetrahedron. Denote its vertices by the points $\mathbf{B}$, $\mathbf{P}^1$, $\mathbf{Q}$, $\mathbf{C}^1$ obtained from two RR chains. The edges $\mathbf{B}-\mathbf{C}^1$ and $\mathbf{P}^1-\mathbf{Q}$ are called *diagonals* of the linkage and have dimensions $2a=|\mathbf{B}-\mathbf{C}^1|$ and $2b=|\mathbf{P}^1-\mathbf{Q}|$, respectively. See Fig. 5.

The axis K of the tetrahedron is defined by the common normal to the diagonals $\mathbf{B}-\mathbf{C}^1$ and $\mathbf{P}^1-\mathbf{Q}$. Let (κ,c) be the dual angle between these edges along K. The axis K is of central importance because the screw axes of the movement of Bennett's linkage are perpendicular to this line. That is, it forms the nodal line of the cylindroid [4,16,17].

Remarkably, the principal axis frame, $\{\mathsf{X},\mathsf{Y},\mathsf{K}\}$, of this cylindroid is located in the midpoint of segment along K joining the diagonals, and oriented such that it bisects the angle κ as it is shown in [18]. Recall that this principal axis frame can be determined directly from the screw axes S_{12} and S_{13} obtained from the three design positions.

We now define the Bennett linkage in the principal axis frame using the four parameters of the tetrahedron a, b, c, κ. The coordinates of the vertices are given by

$$\mathbf{B}=\begin{Bmatrix} a\cos\frac{\kappa}{2}\\ a\sin\frac{\kappa}{2}\\ \frac{-c}{2}\end{Bmatrix},\quad \mathbf{P}^1=\begin{Bmatrix} b\cos\frac{\kappa}{2}\\ -b\sin\frac{\kappa}{2}\\ \frac{c}{2}\end{Bmatrix}, \quad (29)$$

$$\mathbf{Q}=\begin{Bmatrix} -b\cos\frac{\kappa}{2}\\ b\sin\frac{\kappa}{2}\\ \frac{c}{2}\end{Bmatrix},\quad \mathbf{C}^1=\begin{Bmatrix} -a\cos\frac{\kappa}{2}\\ -a\sin\frac{\kappa}{2}\\ \frac{-c}{2}\end{Bmatrix}.$$

To find the direction of the joint axes $\mathbf{G}$ and $\mathbf{W}^1$ we compute the cross product of the edges

$$\mathbf{G}=K_g(\mathbf{Q}-\mathbf{B})\times(\mathbf{P}^1-\mathbf{B})=K_g\begin{Bmatrix} 2bc\sin\frac{\kappa}{2}\\ 2bc\cos\frac{\kappa}{2}\\ 4ab\cos\frac{\kappa}{2}\sin\frac{\kappa}{2}\end{Bmatrix} \quad (30)$$

and

$$\mathbf{W}^1=K_w(\mathbf{B}-\mathbf{P}^1)\times(\mathbf{C}^1-\mathbf{P}^1)=K_w\begin{Bmatrix} -2ac\sin\frac{\kappa}{2}\\ 2ac\cos\frac{\kappa}{2}\\ 4ab\cos\frac{\kappa}{2}\sin\frac{\kappa}{2}\end{Bmatrix}, \quad (31)$$

where the constants K_g and K_w normalize the vectors. The expressions of the screws $\mathsf{G}=(\mathbf{G},\mathbf{B}\times\mathbf{G})^T$ and $\mathsf{W}^1=(\mathbf{W}^1,\mathbf{P}^1\times\mathbf{W}^1)^T$ in the principal axis coordinates are

$$\mathsf{G}=K_g\begin{Bmatrix} 2bc\sin\frac{\kappa}{2}\\ 2bc\cos\frac{\kappa}{2}\\ 4ab\cos\frac{\kappa}{2}\sin\frac{\kappa}{2}\end{Bmatrix} +\epsilon K_g\begin{Bmatrix} b\cos\frac{\kappa}{2}\left(4a^2\sin^2\frac{\kappa}{2}+c^2\right)\\ -b\sin\frac{\kappa}{2}\left(\left(4a^2\cos^2\frac{\kappa}{2}+c^2\right)\right.\\ 2abc\left(\cos^2\frac{\kappa}{2}-\sin^2\frac{\kappa}{2}\right)\end{Bmatrix},$$

and

$$\mathsf{W}^1=K_w\begin{Bmatrix} -2ac\sin\frac{\kappa}{2}\\ 2ac\cos\frac{\kappa}{2}\\ 4ab\cos\frac{\kappa}{2}\sin\frac{\kappa}{2}\end{Bmatrix} +\epsilon K_w\begin{Bmatrix} -a\cos\frac{\kappa}{2}\left(4b^2\sin^2\frac{\kappa}{2}+c^2\right)\\ -a\sin\frac{\kappa}{2}\left(4b^2\cos^2\frac{\kappa}{2}+c^2\right)\\ 2abc\left(\cos^2\frac{\kappa}{2}-\sin^2\frac{\kappa}{2}\right)\end{Bmatrix}. \quad (32)$$

Similarly, the coordinates of the second RR dyad, H and U^1, are given by the expressions

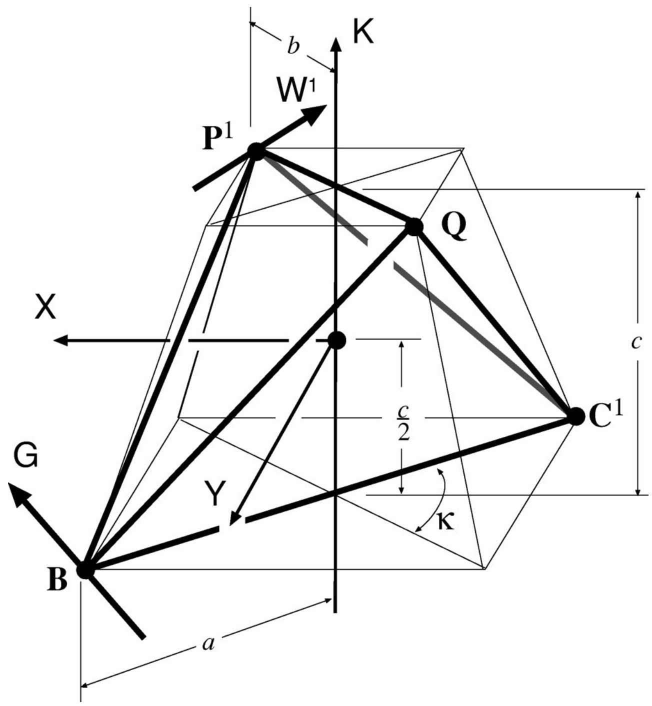

Fig. 5 The tetrahedron defines the Bennett linkage

$$\begin{aligned}\mathsf{H}&=K_h(\mathbf{C}^1-\mathbf{Q})\times(\mathbf{B}-\mathbf{Q})+\epsilon K_h\mathbf{Q}\times((\mathbf{C}^1-\mathbf{Q})\times(\mathbf{B}-\mathbf{Q}))\\ \mathsf{U}^1&=K_u(\mathbf{P}^1-\mathbf{C}^1)\times(\mathbf{Q}-\mathbf{C}^1)+\epsilon K_u\mathbf{C}^1\times((\mathbf{P}^1-\mathbf{C}^1)\times(\mathbf{Q}-\mathbf{C}^1)).\end{aligned}\tag{33}$$

Using these coordinates to define G and W^1, we reduce the number of design parameters from ten to four. Furthermore, the six conditions in Eqs. (27, 28) are identically satisfied. The result is four equations, (25) and (26), in the unknowns a, b, c and κ.

7 Solving the Design Equations

We can transform the task positions $[T_i]$ to the principal axis frame to obtain the relative screw axes in the form

$$\begin{aligned}\mathsf{S}_{12}&=\sin\frac{\hat{\psi}_{12}}{2}(\cos\hat{\delta}_1\mathsf{X}+\sin\hat{\delta}_1\mathsf{Y})\\ \mathsf{S}_{13}&=\sin\frac{\hat{\psi}_{13}}{2}(\cos\hat{\delta}_2\mathsf{X}+\sin\hat{\delta}_2\mathsf{Y}),\end{aligned}\tag{34}$$

where the dual angles $\hat{\delta}_i$ locate these screws in the principal axes frame. Substitute this and the expressions of Eq. (29) into Eq. (26) to obtain two linear equations in the parameters a and b,

$$\frac{t_{12}}{2}+(a-b)\cos\delta_1\cos\frac{\kappa}{2}+(a+b)\sin\delta_1\sin\frac{\kappa}{2}=0$$

$$\frac{t_{13}}{2}+(a-b)\cos\delta_2\cos\frac{\kappa}{2}+(a+b)\sin\delta_2\sin\frac{\kappa}{2}=0.$$

These equations are solved to obtain,

$$\begin{aligned}a&=\frac{K_s}{2\sin\frac{\kappa}{2}}+\frac{K_d}{2\cos\frac{\kappa}{2}},\\ b&=\frac{K_s}{2\sin\frac{\kappa}{2}}-\frac{K_d}{2\cos\frac{\kappa}{2}}.\end{aligned}\tag{35}$$

The constants K_s and K_d are listed in Table 1.

Next we substitute Eq. (35) into the Eq. (25) and define $y=\tan\kappa/2$ to eliminate the sine and cosine functions of κ. The result is two equations in c and y,

$$\frac{\tan\frac{\psi_{12}}{2}\left(\frac{K_s^2}{K_d^2}-y^2\right)+c^2\frac{\tan\frac{\psi_{12}}{2}}{2K_d^2}(y^2(\cos 2\delta_1-1)+\cos 2\delta_1+1)-2\frac{cy}{K_d}\left(\cos\delta_1+\frac{K_s\sin\delta_1}{K_d}\right)}{\left(\frac{K_s^2}{K_d^2}-y^2\right)+\frac{c^2}{2K_d^2}(y^2(\cos 2\delta_1-1)+\cos 2\delta_1+1)}=0,\tag{36}$$

$$\frac{\tan\frac{\psi_{13}}{2}\left(\frac{K_s^2}{K_d^2}-y^2\right)+c^2\frac{\tan\frac{\psi_{13}}{2}}{2K_d^2}(y^2(\cos 2\delta_2-1)+\cos 2\delta_2+1)-2\frac{cy}{K_d}\left(\cos\delta_2+\frac{K_s\sin\delta_2}{K_d}\right)}{\left(\frac{K_s^2}{K_d^2}-y^2\right)+\frac{c^2}{2K_d^2}(y^2(\cos 2\delta_2-1)+\cos 2\delta_2+1)}=0.\tag{37}$$

The numerator and denominator of these equations share two roots associated with $c=0$, that is,

$$(c,y)=\left(0,\pm\frac{K_s}{K_d}\right).\tag{38}$$

In order to solve Eqs. (36), (37), we eliminate the roots (38) by requiring nontrivial solutions to the linear system

$$\begin{bmatrix} T\frac{\psi_{12}}{2} & \frac{T\frac{\psi_{12}}{2}}{2K_d^2}(y^2c(C2\delta_1-1)+C2\delta_1+1)-2\frac{y}{K_d}\left(C\delta_1+\frac{K_sS\delta_1}{K_d}\right)\\ T\frac{\psi_{13}}{2} & \frac{T\frac{\psi_{13}}{2}}{2K_d^2}(y^2c(C2\delta_2-1)+C2\delta_2+1)-2\frac{y}{K_d}\left(C\delta_2+\frac{K_sS\delta_2}{K_d}\right)\end{bmatrix}\begin{Bmatrix}K_s^2/K_d^2-y^2\\ c\end{Bmatrix}=0,\tag{39}$$

Table 1 Constants computed from the specified positions

Constant	Expression
K_s	$\dfrac{t_{12}\cos\delta_2 - t_{13}\cos\delta_1}{2\sin(\delta_1-\delta_2)}$
K_d	$\dfrac{t_{13}\sin\delta_1 - t_{12}\sin\delta_2}{2\sin(\delta_1-\delta_2)}$
K_{12}	$\dfrac{t_{12}/2}{\tan\dfrac{\psi_{12}}{2}}\left(\dfrac{1}{\sin^2\delta_1-\sin^2\delta_2}\right)$
K_{13}	$\dfrac{t_{13}/2}{\tan\dfrac{\psi_{13}}{2}}\left(\dfrac{1}{\sin^2\delta_1-\sin^2\delta_2}\right)$

Table 2 The two sets of solutions

Solution 1		Solution 2
$\mathsf{G}(a,b,c,\kappa)$	=	$\mathsf{H}(-b,-a,-c,-\kappa)$
$\mathsf{W}^1(a,b,c,\kappa)$	=	$\mathsf{U}^1(-b,-a,-c,-\kappa)$
$\mathsf{H}(a,b,c,\kappa)$	=	$\mathsf{G}(-b,-a,-c,-\kappa)$
$\mathsf{U}^1(a,b,c,\kappa)$	=	$\mathsf{W}^1(-b,-a,-c,-\kappa)$

where the capital letters C, S, T stand for cosine, sine and tangent respectively. This is achieved by requiring that the coefficient matrix have the determinant zero. The result is an equation that is linear in c, which yields

$$c=(K_{13}-K_{12})\sin\kappa, \tag{40}$$

where the constants K_{12} and K_{13} are shown in Table 1.

In order to determine κ, we substitute the expressions for a, b and c into one of the direction equations in Eq. (25). The result is a cubic polynomial in y^2,

$$P:\quad C_3y^6+C_2y^4+C_1y^2+C_0=0 \tag{41}$$

with the coefficients

$$\begin{aligned} C_3&=-K_d^2,\\ C_2&=K_s^2-2K_d^2+4(K_{12}-K_{13})(K_{13}\sin^2\delta_1-K_{12}\sin^2\delta_2),\\ C_1&=2K_s^2-K_d^2-4(K_{12}-K_{13})(K_{13}\cos^2\delta_1-K_{12}\cos^2\delta_2),\\ C_0&=K_s^2. \end{aligned} \tag{42}$$

Substitute $x=y^2$, and solve the cubic polynomial to determine its three roots.

We can show that the polynomial $P(x)$ has only one positive real root for x, which we denote x^*. To do this we compute the values,

$$\begin{aligned} P(0)&=K_s^2,\\ P(-1)&=-4(K_{12}-K_{13})^2,\\ P(+x_\infty)&\approx-K_d^2(+x_\infty)^3,\\ P(-x_\infty)&\approx-K_d^2(-x_\infty)^3. \end{aligned} \tag{43}$$

Notice that $P(0)>0$, while for large positive values of x we have $P(+x_\infty)<0$. From this we conclude that the polynomial has at least one positive real root. Furthermore, $P(-1)<0$ for all values of its coefficients, and for large negative values of x the polynomial is positive, $P(-x_\infty)>0$. This leads to the conclusion that there are at least two negative real roots. The cubic polynomial only has three roots, therefore one is positive and two are negative.

Table 3 Three specified positions

	x	y	z	θ	ϕ	ψ
M_1	0.0	0.0	0.0	0 deg	0 deg	0 deg
M_2	0.0	0.0	0.8	0 deg	0 deg	40 deg
M_3	1.11	0.66	0.05	18.8 deg	−28.0 deg	67.2 deg

To determine κ, we compute the square root of the single positive real root x^*, which yields the two solutions,

$$\kappa=2\arctan(\pm\sqrt{x^*}). \tag{44}$$

Substitute this into the formulas (35) and (40) and we obtain the pair of solutions (a,b,c,κ) and $(-b,-a,-c,-\kappa)$. Evaluating the two RR chains obtained for these solutions (32) and (33), we obtain the results in Table 2. The symmetry of this result shows that these solutions necessarily define a Bennett linkage.

8 Example

In Table 3 and Fig. 6 we present three goal positions for the coupler of a Bennett linkage. They are defined in terms of a vector (x,y,z) and the longitude, latitude and roll angles (θ,ϕ,ψ) that define the locations for the moving frame.

Solving the design equations, we obtain the fixed joint axes G and H and moving joint axes W^1 and U^1. Table 4 presents their coordinates in the original frame. Figure 7 shows the resulting Bennett linkage in each of the goal positions.

9 Conclusions

This paper presents a new formulation of the design equations for three position synthesis of a Bennett linkage. The procedure

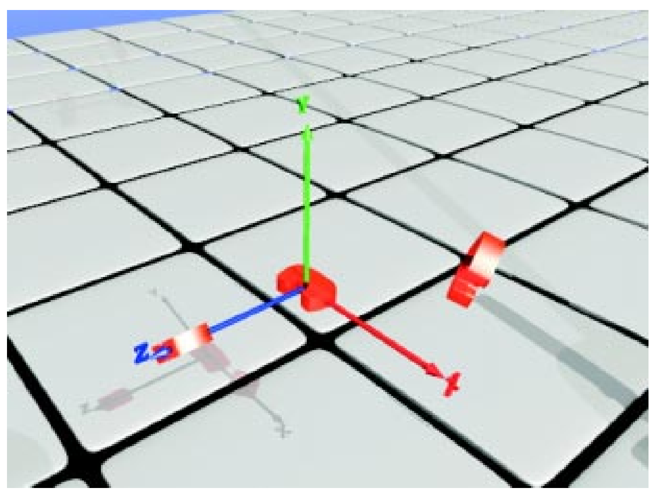

Fig. 6 The specified positions

Table 4 The joint axes of the Bennett linkage

Axis	Line coordinates
G	(0.36,0.45,0.81), (0.26,1.05,−0.70)
W^1	(0.60,0.36,0.72), (0.87,0.83,−1.14)
H	(0.60,−0.36,0.72), (0.87,−0.83,−1.14)
U^1	(0.36,−0.45,0.81), (0.26,−1.05,−0.70)

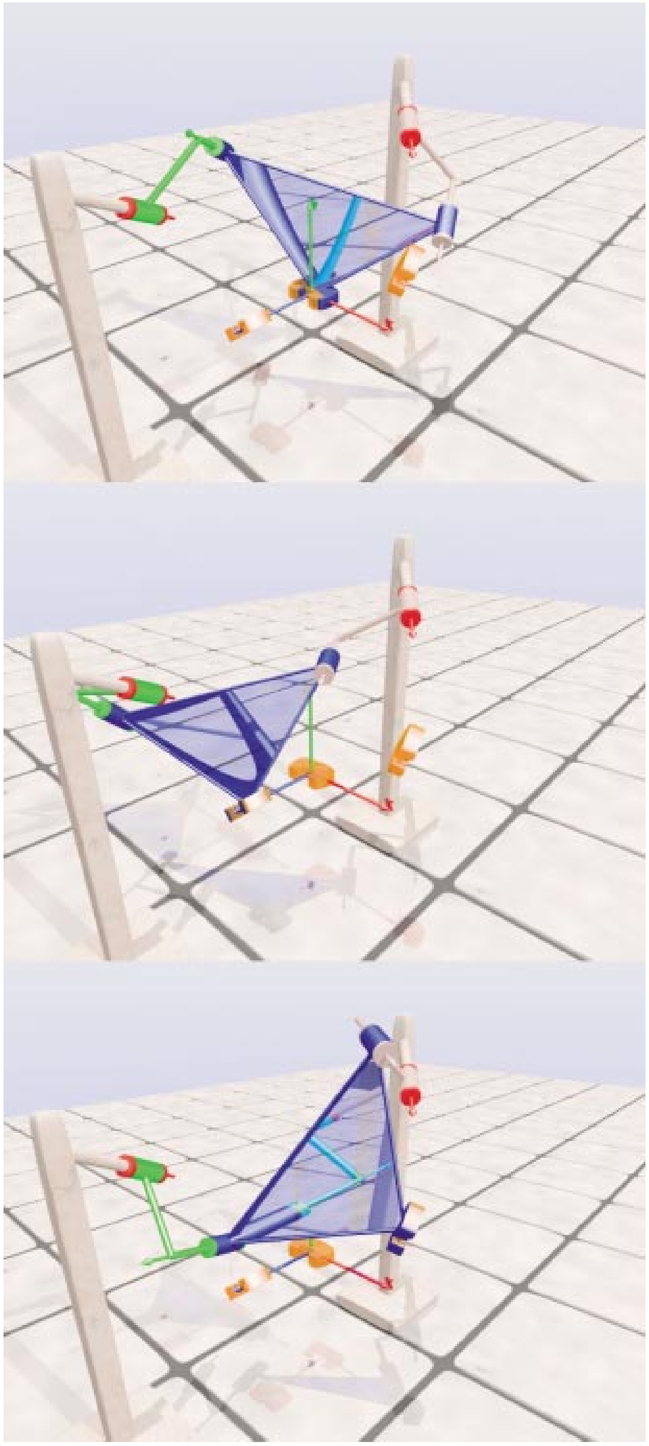

Fig. 7 The Bennett linkage at the three specified positions

combines the results of Tsai and Roth [3] for the spatial RR chain with the geometric properties of the cylindroid studied by Huang [4]. Introduction of the principal axis frame of this cylindroid and a set of design parameters adapted to Bennett's linkage reduces the formulation from ten quadratic equations in ten unknowns to four equations in four unknowns. We show that the resulting equations have two real solutions that must form a Bennett linkage.

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