Geometric Design of Cylindric PRS Serial Chains

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Abstract

This paper examines the design equations for the cylindric PRS serial chain. This five degree-of-freedom robot can be designed to reach an arbitrary set of eight spatial positions. However, it is often convenient to choose some of the design parameters and specify a task with fewer positions. For this reason, we study the three through eight position synthesis problems and consider various choices of design parameters for each. A linear product decomposition is used to obtain bounds on the number of solutions to these design problems. For all cases of six or fewer positions, the bound is exact and we give a reduction of the problem to the solution of an eigenvalue problem. For seven and eight position tasks, the linear product decomposition is useful for generating a start system for solving the problems by continuation. The large number of solutions so obtained contraindicates an elimination approach for seven or eight position tasks, hence continuation is the preferred approach.

1 Introduction

This paper examines the geometric design of a five degree-of-freedom PRS serial chain constructed so that the prismatic, or sliding, joint (P) and the revolute,

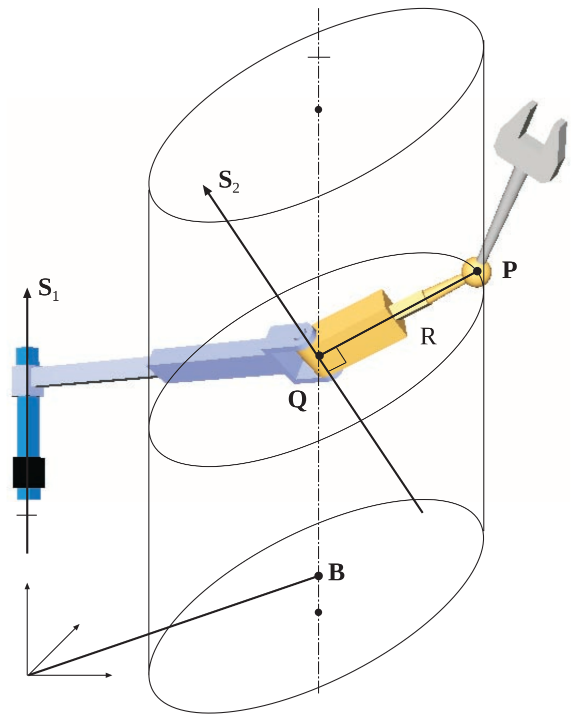

Figure 1: The PRS serial chain

or hinged, joint (R) are parallel. In this configuration the center of the spherical wrist (S) moves on a right circular cylinder around the axis of the revolute joint. We call this a “cylindric” RPS chain. Our goal is to determine the dimensions and location of this cylinder so that an end-effector of the chain can reach an arbitrary set of task positions.

A cylindric PRS chain is kinematically equivalent to the CS chain, where C denotes a cylindric joint. Our choice of terminology emphasizes the fact that this chain is a special case of the general PRS chain in which the S joint lies on a skew cylinder that is traced by a circle that is tilted relative to the direction of travel along the prismatic joint. See Figure 1.

2 Literature Review

The elementary principles of the geometric design of linkages can be found in the text by McCarthy (2000). An advanced approach by Tsai (1973) provides a foundation for the design of robotic systems; also see Tsai and Roth (1972). Krovi et al. (2001) study the design of coupled spatial RR chains, Liao and McCarthy (2001) focus on SS chains and their assembly into the single degree-of-freedom 5-SS platform, Mavroidis et al. (2001) obtain design equations for spatial RR chains using robotic kinematics equations and Lee and Mavroidis (2002) solved several design problems for RRR chains by polynomial continuation. Kihonge et al. (2002) provide virtual reality design environment for CC chains, and 4C linkages. Current research is focussed on generalizing these ideas to achieve task-based design for serial chains with two to five degrees of freedom.

This paper addresses the design of cylindric PRS chains, which, having the directions of the prismatic and revolute joints equal, may also be called a CS chain. The design equations for a cylindric PRS chain were studied originally by Chen and Roth (1967). They considered the design of systems for which the direction of the cylindric joint is specified by the designer and six task positions are specified. They concluded that this design problem had at most 26 solutions. A solution procedure for these design equations was presented by Nielsen and Roth (1995) using sparse matrix elimination techniques. This approach yields 175 sparse equations in 140 monomials that can be reduced to a 26×26 generalized eigenvalue problem.

In addition to the six-position problem considered in the literature, we also solve cases where from three up to the maximum of eight task positions are specified. For each of these, we consider various possibilities for specifying a subset of design parameters to exactly determine the design. (That is, the total of the number of task positions and the number of specified design parameters is eight.) This allows a designer the flexibility to trade away direct control of the design parameters to obtain more task positions or vice versa.

For all of these design problems, we show how to bound the number of solutions using a linear product decomposition. This counting method is equivalent to the "set structure" theory in Verschelde and Haegemans (1993) and is a special case of the general product decomposition theory presented in Morgan et al. (1995). For six or fewer task positions, this count is exact. Since the solution count is manageable for these cases (at most 26), we give elimination procedures for each. These procedures resemble the one in Nielsen and Roth (1995) and reduce to solving a generalized eigenvalue problem.

In order to analyze the seven and eight position design problems, we use numerical polynomial continuation, specifically the public-domain software PHC developed by Verschelde (1999). The first use of continuation to synthesize a spatial chain was the treatment of the seven-position SS chain in Wampler et al (1990). More recently, Lee and Mavroidis (2002) used the method for synthesizing RRR chains.

The synthesis problems solved here can be used to design a variety of mechanisms. These can range from a single RPS chain which is actuated as an open-chain, five-degree-of-freedom robot, to a one-degree-of-freedom spatial mechanism having five PRS legs in parallel. For example, three PRS chains are used as the legs of the Eclipse parallel machining center described in Ryu et al. (1998) and in Kim et al. (1999). In this case, the axes of the P and R joints are at right angles, and the skew cylinder flattens into a plane parallel to the axis of the P joint.

3 The PRS serial chain

Let the axes of the P and R joints of the general PRS serial chain be defined by the Plucker vectors $\mathsf{S}_1 = (\mathbf{S}_1, \mathbf{c}_1 \times \mathbf{S}_1)$ and $\mathsf{S}_2 = (\mathbf{S}_2, \mathbf{c}_2 \times \mathbf{S}_2)$. Recall that $\mathbf{S}_i$ is the direction of the joint axis and $\mathbf{c}_i$ is a point on this axis. The common normal to these axes A_{12} allows us to define the distance a and angle α between these lines along and around A_{12}.

Denote the center of the spherical wrist as $\mathbf{P}$, and let $\mathbf{Q}$ be the point on S_2 closest to $\mathbf{P}$. Notice that $\mathbf{Q}$ moves along a line G parallel to the prismatic axis S_1, while $\mathbf{P}$ moves on a circle about $\mathbf{P}$ in a plane perpendicular to $\mathbf{S}_2$. Let $\mathbf{B}$ be a fixed point that lies on G such that the variable distance between it and $\mathbf{Q}$ is k. For convenience, we now introduce the unit vectors $\vec{\imath} = (\mathbf{S}_2 \times \mathbf{S}_1)/\sin\alpha$ and $\vec{\jmath} = \mathbf{S}_2 \times \vec{\imath}$. These definitions allow us to define the location of $\mathbf{P}$ relative to $\mathbf{B}$ as

$$\mathbf{P} - \mathbf{B} = k\mathbf{S}_1 + R(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath}), \tag{1}$$

where $R = |\mathbf{P} - \mathbf{Q}|$ and θ is the angle measured around $\mathbf{S}_2$ from $\vec{\imath}$.

It is now easy to compute the equation

$$\mathbf{S}_2 \times \big((\mathbf{P} - \mathbf{B}) \times \mathbf{S}_1\big) = R\cos\alpha(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath}), \tag{2}$$

in which we have used the identity $\mathbf{S}_1 = \cos\alpha\mathbf{S}_2 - \sin\alpha\vec{\jmath}$. Thus, we obtain the constraint equation for the PRS chain as

$$\big(\mathbf{S}_2 \times ((\mathbf{P} - \mathbf{B}) \times \mathbf{S}_1)\big)^2 = R^2(\mathbf{S}_1 \cdot \mathbf{S}_2)^2. \tag{3}$$

This equation has 13 dimensional parameters: the radius R, three each for the directions $\mathbf{S}_1$, $\mathbf{S}_2$, and three each for $\mathbf{P}$ and $\mathbf{B}$. However, there are actually only 10 independent parameters available for design purposes. In what follows, we determine three additional linear constraints.

First, we note that any point along the line G can serve as the reference point $\mathbf{B}$ for the axis of the cylinder. We determine $\mathbf{B}$ by specifing an arbitrary plane $U : (\mathbf{n}, d)$. In general, the line G must intersect this plane, and we select this point at $\mathbf{B}$. Thus, $\mathbf{B}$ satisfies the linear equation

$$\mathbf{n} \cdot \mathbf{B} = d. \tag{4}$$

Notice that $\mathbf{n}$ is the unit normal to the plane and d the directed distance from the origin to the plane.

Next, we note that it is the directions of $\mathbf{S}_1$ and $\mathbf{S}_2$ that matter not their magnitude. The vectors $\mathbf{S}_1$ and $\lambda\mathbf{S}_1$, for λ a nonzero scalar, define the same mechanism, and similarly for $\mathbf{S}_2$. It is not recommended that this ambiguity be resolved by requiring $\mathbf{S}_i$ to be unit vectors. This results in a duplication of solutions, because $\pm\mathbf{S}_i$ both satisfy this condition. We introduce the two arbitrary planes $V_j : (\mathbf{m}_j, e_j), j = 1, 2$, and note again that in general the lines S_i must intersect these planes, respectively. We select these points of intersection as $\mathbf{S}_i$, such that they satisfy the linear constraints,

$$\mathbf{m}_j \cdot \mathbf{S}_j = e_j, \qquad j = 1, 2. \tag{5}$$

In order to design a general PRS chain we assume that we are given n goal positions defined by the transformations $[T_i], i = 1, \ldots, n$. We then seek a point $\mathbf{p}$ in the moving body that $\mathbf{P}^i = [T_i]\mathbf{p}$ satisfies the constraint equation (3) in each of the goal positions. The general solution for 10 goal positions yields the following system of polynomials:

$$\begin{aligned}\big(\mathbf{S}_2 \times \big((\mathbf{P}^i - \mathbf{B}) \times \mathbf{S}_1\big)\big)^2 - \big(\mathbf{S}_2 \times \big((\mathbf{P}^1 - \mathbf{B}) \times \mathbf{S}_1\big)\big)^2 = 0, \quad i = 2, \ldots, 10,\\ \mathbf{n} \cdot \mathbf{B} = d, \quad \text{and} \quad \mathbf{m}_j \cdot \mathbf{S}_j = e_j,\ j = 1, 2.\end{aligned} \tag{6}$$

Note that we have subtracted the first constraint equation from the remaining nine in order to eliminate the term $R^2(\mathbf{S}_1 \cdot \mathbf{S}_2)^2$. The result is 12 polynomial equations in 12 unknowns.

The system (6) of nine sixth degree and three linear polynomials has a total degree of $6^9 \approx 10^7$. This system may be solvable using an appropriate multihomogeneous formulation or other more advanced homotopies, however, it will be an expensive calculation. Rather than pursue this, we simplify the problem by requiring that the prismatic and revolute joints be parallel.

4 The cylindric PRS chain

If we require the axis of the prismatic joint to be parallel to that of the revolute joint, that is $\mathbf{S}_1 = \mathbf{S}_2 = \mathbf{G}$, three components of the vector $\mathbf{S}_2$ and the constraint equation $\mathbf{m}_2 \cdot \mathbf{S}_2 = e_2$ are eliminated. The result is a loss of two design parameters. Thus, a cylindric PRS chain can be designed to reach at most eight spatial positions.

We now use the identity $(\mathbf{a}\times(\mathbf{b}\times\mathbf{a}))^2 = (\mathbf{b}\times\mathbf{a})^2\mathbf{a}^2$ to simplify the constraint equation (6) and obtain

$$\big((\mathbf{P}^i - \mathbf{B}) \times \mathbf{G}\big)^2 - \big((\mathbf{P}^1 - \mathbf{B}) \times \mathbf{G}\big)^2 = 0, \tag{7}$$

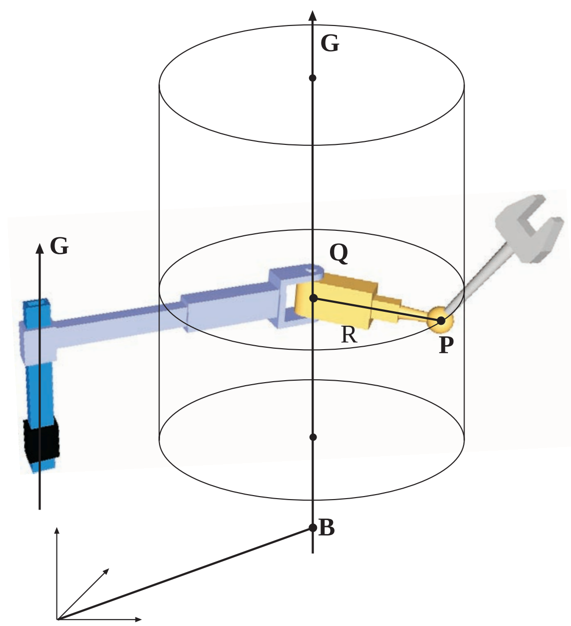

Figure 2: The cylindric PRS serial chain.

where we have concelled the $\mathbf{G}^2$ term. Expand this further and cancel $(\mathbf{B}\times\mathbf{G})^2$, so we obtain the system of design equations for eight task positions,

$$\begin{aligned}
&\mathcal{P}_i: \quad (\mathbf{P}^i \times \mathbf{G})^2 - (\mathbf{P}^1 \times \mathbf{G})^2 + 2[(\mathbf{P}^1 - \mathbf{P}^i) \times \mathbf{G})] \cdot (\mathbf{B} \times \mathbf{G}) = 0,\ i = 2, \dots, n,\\
&\mathcal{C}_1: \quad \mathbf{n}\cdot\mathbf{B} = d \quad \text{and} \quad \mathcal{C}_2: \quad \mathbf{m}\cdot\mathbf{G} = e. \qquad (8)
\end{aligned}$$

Recall that $\mathbf{P}^i = [T_i]\mathbf{p}$. Each solution of this set of seven fourth degree polynomials $\mathcal{P}_i$, and two linear equations $\mathcal{C}_1$ and $\mathcal{C}_2$ defines a cylindric PRS serial chain that reaches the specified task positions.

Notice that we can specify a set of the dimensional parameters, and thereby reduce the number of task positions for the design problem. This allows the designer to trade control of the task against a direct influence on the mechanism's geometry. In particular, if the designer chooses to specify j dimensional parameters, then the associated task would have $n = 8 - j$ positions.

To tabulate the possibilities, let $g \leq 2$, $p \leq 3$, and $b \leq 2$ be the number of components specified for $\mathbf{G}$, $\mathbf{p}$, and $\mathbf{B}$, respectively. Recall, $n + g + p + b = 8$. Following Chen and Roth (1967), we specify $\mathbf{G}$ when possible, which means we have $g = 2$ for all of $n \leq 6$. Table 1 shows the combinations of specified parameters and tasks that are available.

Name	n	g	p	b
CS3a	3	2	3	0
CS3b	3	2	2	1
CS3c	3	2	1	2
CS4a	4	2	2	0
CS4b	4	2	1	1
CS4c	4	2	0	2
CS5a	5	2	1	0
CS5b	5	2	0	1
CS6	6	2	0	0
CS7a	7	1	0	0
CS7b	7	0	1	0
CS7c	7	0	0	1
CS8	8	0	0	0

Table 1: CSn Design Problems

5 Counting Solutions

For a system of n polynomial equations in n variables, a result known as Bezout's theorem states that the number of solutions must be less than or equal to the product of the degrees d_i of the polynomials, that is $D = d_1 d_2 \dots d_n$, which is called the total degree of the system.

More refined estimates of the number are possible by considering the strucuture of these polynomials. To do this we compile a list of the monomials that appear in the i^{th} polynomial of the system. The collection of n lists of monomials forms the "monomial structure" of the system. It is a fundamental result in Algebraic Geometry that polynomial systems that have the same monomial structure, and differ only in the scalar coefficients of each of the monomials, form a family for which almost all have the same number of solutions over the complex numbers. We call this the "root count" or the generalized "Bezout number" of the family. Exceptional members of the family cannot have more than this number of isolated solutions (although they may have curves or other higher dimensional solution sets); that is, the Bezout number is a bound on the number of isolated solutions, or roots.

Each of our design problems has a particular monomial structure with its coefficients determined by the task positions and whatever design parameters we have specified. In what follows, we determine bounds on the number of roots for each case.

5.1 Linear Product Decomposition

Bernshtein (1975) shows that the root count for any monomial structure can be obtained from the mixed volume of the associated Newton polytopes, a result sometimes known as the "BKK bound." This mixed volume is combinatorial in nature, and can require computer calculation. However, a special case known as the linear product decomposition bound is applicable to our cylindric PRS design problem, and is convenient for hand calculation.

A monomial list has a product decomposition (Morgan et al, 1995) if each monomial in this list can be obtained as a product of one element from each of two or more "factor" lists of monomials. If the monomials in these factor lists have degree at most one, then the product decomposition is *linear*, also called a set structure by Verschelde and Haegemans (1993). The important result is that a root count generated from the product decomposition is the same as that for the original list.

The linear product decomposition allows the determination of the number of roots by tabulation of all ways of choosing one linear factor from each equation, such that all the variables are determined. Each admissible set of linear factors defines a single root. It is not necessary that each equation have the same monomial structure, and any number of linear factors can be present.

As an example, consider the pair of quadratic equations both of which have the monomial structure $\langle 1, x, y, x^2, xy \rangle$,

$$\begin{aligned} c_{11} + c_{12}x + c_{13}y + c_{14}x^2 + c_{15}xy = 0, \\ c_{21} + c_{22}x + c_{23}y + c_{24}x^2 + c_{25}xy = 0, \end{aligned} \tag{9}$$

where c_{ij} are non-zero scalar coefficients. This monomial structure has the linear product decomposition $\langle 1, x, y, x^2, xy \rangle = \langle 1, x \rangle \langle 1, x, y \rangle$, therefore the pair of equations

$$\begin{aligned} (d_{11} + d_{12}x)(d_{13} + d_{14}x + d_{15}y) = 0 \\ (d_{21} + d_{22}x)(d_{23} + d_{24}x + d_{25}y) = 0, \end{aligned} \tag{10}$$

with d_{ij} as non-zero coefficients, will have the same number of roots as (9).

We can see that while there are four ways to choose a linear factor from the two equations in (10), only three allow us to compute both x and y. The set consisting of two linear terms in x is not admissible, because it cannot be used to determine y. Thus, the system Eq.(10) has a Bezout number of three, and by the product decomposition theory, so does system Eq.(9).

It useful to note that if a polynomial in n variables contains all of the available monomials, then its monomial structure has the linear product decomposition $\langle 1, x_1, \ldots, x_n \rangle^d$, where d is its degree. The Bezout number computed using the linear product decomposition for n such polynomials, in which the i^{th} polynomial has degrees d_i, is the total degree $D = d_1 d_2 \ldots d_n$.

Returning to our example, two general quadratic polynomial in two variables have the monomial structure $\langle 1, x, y, x^2, xy, y^2\rangle = \langle 1, x, y\rangle^2$ and has four roots. We now see that the reduction to three roots results from the fact that y^2 is missing in the two equations.

5.2 Root Counts for CS-n Designs

We now consider the monomial structure of our design equations (8) which has the design variables $\mathbf{G} = (g_1, g_2, g_3)$, $\mathbf{B} = (b_1, b_2, b_3)$ and $\mathbf{p} = (p_1, p_2, p_3)$. The polynomials $\mathcal{P}_i$ are linear combinations of monomials in the set generated by

$$(\langle g_1, g_2, g_3\rangle\langle 1, p_1, p_2, p_3\rangle)^2 + \langle g_1, g_2, g_3\rangle^2\langle 1, p_1, p_2, p_3\rangle\langle b_1, b_2, b_3\rangle. \qquad (11)$$

Here we have used the fact that $\mathbf{P}^i = [T_i]\mathbf{p}$. Notice that $(\langle g_1, g_2, g_3\rangle\langle 1, p_1, p_2, p_3\rangle)^2 \subset \langle g_1, g_2, g_3\rangle^2\langle 1, p_1, p_2, p_3\rangle^2$, therefore (11) becomes

$$\langle g_1, g_2, g_3\rangle^2(\langle 1, p_1, p_2, p_3\rangle^2 + \langle 1, p_1, p_2, p_3\rangle\langle b_1, b_2, b_3\rangle), \qquad (12)$$

which can be written as

$$\langle g_1, g_2, g_3\rangle^2\langle 1, p_1, p_2, p_3\rangle\langle 1, p_1, p_2, p_3, b_1, b_2, b_3\rangle, \qquad (13)$$

since $(\langle 1, p_1, p_2, p_3\rangle + \langle b_1, b_2, b_3\rangle) \subset \langle 1, p_1, p_2, p_3, b_1, b_2, b_3\rangle$.

This shows that the design equations (8) have the monomial structure,

$$\begin{aligned} \mathcal{P}_i \in &\quad \langle g_1, g_2, g_3\rangle^2\langle 1, p_1, p_2, p_3\rangle\langle 1, p_1, p_2, p_3, b_1, b_2, b_3\rangle = 0,\ i = 2, \dots, n, \\ \mathcal{C}_1 \in &\quad \langle b_1, b_2, b_3, 1\rangle = 0, \quad \text{and} \\ \mathcal{C}_2 \in &\quad \langle g_1, g_2, g_3, 1\rangle = 0. \end{aligned} \qquad (14)$$

Notice that the constraint equations $\mathcal{P}_i$ are of degree four when $\mathbf{G}$ is an unknown, but become quadratic if $\mathbf{G}$ is specified.

5.2.1 $3 \leq n \leq 6$ Task positions

We now enumerate the root count for the tasks CSn, with $3 \leq n \leq 6$. For these cases, we assume $\mathbf{G}$ is specified so the monomial structure of the design equations becomes

$$\begin{aligned} \mathcal{P}_i \in &\quad \langle 1, p_1, p_2, p_3\rangle\langle 1, p_1, p_2, p_3, b_1, b_2, b_3\rangle = 0,\ i = 1, \dots, n-1 \leq 5, \\ \mathcal{C}_1 \in &\quad \langle b_1, b_2, b_3, 1\rangle = 0. \end{aligned} \qquad (15)$$

If less that six task positions are specified then we can have p specified components of $\mathbf{p}$, and b specified components of $\mathbf{B}$, such that $6 - n = p + b$. Thus, if n and p are given, then b is fixed.

Each root of (15) is associated with an admissible set of linear terms that define the variables. For the first $n-1$ equations, we can select at most $3-p$ of the factors $\langle 1, p_1, p_2, p_3 \rangle$, because more that this would would overdetermine the variables in $\mathbf{p}$. The remaining terms must be taken from the second factor. We have no option in the selection of the factor from the last equation. These facts yield a formula for the linear decomposition bound for n task positions and p components specified, given by

$$\mathrm{LPD}(n,p) = \sum_{i=0}^{3-p} \binom{n-1}{i}, \quad n \leq 6. \tag{16}$$

Table 2 lists the values given by this formula for the various design problems. The entry $D = 2^{n-1}$ is the total degree of the system—this becomes one for CS3a because $\mathbf{p}$ is completely specified. Notice that for problem CS6, the LPD bound of 26 is equal to the root count given previously by Chen and Roth (1967).

Name	CS3a	CS3b	CS3c	CS4a	CS4b	CS4c	CS5a	CS5b	CS6
n	3	3	3	4	4	4	5	5	6
p	3	2	1	2	1	0	1	0	0
D	1	4	4	8	8	8	16	16	32
LPD	1	3	4	4	7	8	11	15	26

Table 2: Linear-product bounds for $n \leq 6$.

5.2.2 7 and 8 Task positions

For $n = 7$ and $n = 8$ task positions, we must include the direction $\mathbf{G}$ in the variable list which means the contraint equations are of fourth degree. The linear product structure (14) allows us to compute a bound on the number of roots to these two design problems. Notice that for CS7 we can specify only one component of $\mathbf{G}$, $\mathbf{p}$, or $\mathbf{B}$, and that for CS8 none can be specified. We consider the case of CS8 first.

We must choose two terms from the first factor $\langle g_1, g_2, g_3 \rangle$ in $\mathcal{P}_i$ which combine with $\mathcal{C}_2$ to define a root $\mathbf{G}$. Because this factor is squared, the number of choices is increased by a factor of $2^2 = 4$. Next we can choose up to three terms from second factor $\langle p_1, p_2, p_3, 1 \rangle$ in the five remaining constraint equations to define $\mathbf{p}$. Any remaining terms must come from the third factor. This yields

$$\mathrm{LPD}(n,g,p) = \mathrm{LPD}(8,0,0) = 2^2 \binom{7}{2} \sum_{i=0}^{3} \binom{5}{i} = 2184, \tag{17}$$

which is much reduced from the total degree of $4^7 = 16384$.

This formula can be generalize to include the case CS7 by noting that $8-n = g+p+b$, where g is the number of components specified in g, and p and b the number of components specified in $\mathbf{p}$ and $\mathbf{B}$, as described above. Using this notation, the number of choices for the terms taken from the first factor is up to $2-g$ taken from $n-1$ equations, which is multiplied by 2^{2-g} to take into account the multiplicity of this factor. Finally, the number of terms taken from the second factor is up to $3-p$ factors taken from $(n-1)-(2-g)$ equations. The remainder of the terms are taken from the third factor. Thus, we have the formula for the root count for cases CS7 and CS8 as

$$\mathrm{LPD}(n,g,p) = 2^{2-g}\binom{n-1}{2-g}\sum_{i=0}^{3-p}\binom{n+g-3}{i}, \quad n=7,8. \tag{18}$$

The values of this formula for the various designs is shown in Table 3.

Name	CS7a	CS7b	CS7c	CS8
n	7	7	7	8
g	1	0	0	0
p	0	1	0	0
Deg	4096	4096	4096	16384
LPD	312	660	900	2184

Table 3: Linear-product bounds for $n = 7, 8$.

6 Solution by Elimination

For $3 \le n \le 6$ task positions, the LPD bound is low enough to suggest that a variable elimination procedure may be convenient. In what follows, we describe the mathematical framework for our elimination procedure, and then apply it to these design problems.

6.1 Eigenvalue Elimination Procedure

Wampler (2002) shows how to formulate the solution of the kinematics equations of multi-loop spherical linkages as a generalized eigenvalue problem. We apply this technique to our design equations. The procedure has the following basic steps:

1. Consider the set of m polynomials $\mathcal{P}_i$, in m variables x_i, $i = 1, \dots, m$. Choose k of these variables and form the list of monomials $M = \binom{k+\mu}{\mu}$ up

to degere μ. Multiply the given equations by this list to define the set of Mm polynomials $\mathcal{Q}_j$. Now each of these polynomials can be written as a linear combination of the monomials in a list $\mathbf{y}$ of N monomials, such that $\mathcal{Q}_j = \Sigma a_{jl} y_l$. This set of polynomials can be written in the form $A\mathbf{y} = 0$, where A is an $Mm \times N$ constant matrix.

2. Gaussian elimination of A generates a row reduced set of r=rank[A] independent equations $B\mathbf{y} = 0$, where B is an $r \times N$ constant matrix.

3. Select one of the variables x_i to be the suppressed variable λ, say $x_1 = \lambda$, and generate identities of the form $y_i - \lambda y_j = 0$, where y_i and y_j are monomials in the list $\mathbf{y}$. For example, the monomials $x_1 x_2$ and $x_1^2 x_2 = \lambda x_1 x_2$ satisfy the identify $(1)x_1^2 x_2 - (\lambda)x_1 x_2 = 0$.

4. Append $N - r$ of these identities $[\lambda C + D]\mathbf{y} = 0$, where both C and D have entries that are simply 1 or -1, in order to define the $N \times N$ matrix equation

$$[E(\lambda)]\mathbf{y} = \begin{bmatrix} B \\ \lambda C + D \end{bmatrix} \mathbf{y} = 0. \tag{19}$$

5. If the matrix $E(\lambda)$ has full rank for arbitrary values of λ, then the $N - r$ values λ defined by $\det[E(\lambda)] = 0$ are its generalized eigenvalues. The solutions for x_1 to the original set of polynomials $\mathcal{P}_i$ must be among these eigenvalues, and the associated eigenvector defines the values of the remaining variables x_i. The matrix $N \times N$ determinant $\det E$ can be reduced to an $N - r \times N - r$ eigenvalue probem, see Appendix.

This process yields the roots of the origninal set of polynomials, however, it may also generate extraneous roots. In particular, an eigenvalue can yield an eigenvector that has a zero coefficient for the monomial 1; this is equivalent to a solution at infinity.

While this procedure is general, there are three related aspects that must be adapted to a given set of polynomials: (i) the selection of monomials in Step 1, (ii) the choice of the suppressed variable, and (iii) the selection of the identities in identities in Step 4. The rank of $E(\lambda)$ is a convenient test whether the monomials and identities have been defined completely. However, once this is done the formulation is valid for a general case in the family polynomials systems related to $\mathcal{P}_i, i = 1, \dots, m$.

6.2 Application to CSn, $3 \leq n \leq 6$

The monomial structure of the CSn problems is given in (15). Recall that p and b denote the number of components, respectively, of $\mathbf{p}$ and $\mathbf{B}$ that are

Table 4: Elimination formulations for Tasks $3 \leq n \leq 6$

Case	Multipliers	Mm	λ	Identities	N	r	$N-r$
CS3a	$\langle 1 \rangle$	2	b_1	$\langle 1 \rangle$	3	2	1
CS3b	$\langle 1 \rangle$	2	p_1	$\langle 1, p_1, b_1 \rangle$	5	2	3
CS3c	$\langle 1, p_1, p_2 \rangle$	6	p_1	$\langle 1, p_1, p_2, p_1^2 \rangle$	10	6	4
CS4a	$\langle 1 \rangle$	3	p_1	$\langle 1, p_1, b_1, b_2 \rangle$	7	3	4
CS4b	$\langle 1, p_1, p_2 \rangle^2$	18	b_1	$\langle 1, p_1, p_1^2, p_1^3 \rangle \cup \langle p_2, p_1 p_2, p_1^2 p_2 \rangle$	25	18	7
CS4c	$\langle 1, p_1, p_2, p_3 \rangle^2$	30	p_1	$\langle 1, p_1, p_2 \rangle^2 \cup \langle p_1^3, p_1^2 p_2 \rangle$	35	27	8
CS5a	$\langle 1, p_1, p_2 \rangle^2$	24	b_1	$\langle 1, p_1, p_2 \rangle^3 \cup \langle p_1^4 \rangle$	35	24	11
CS5b	$\langle 1, p_1, p_2, p_3 \rangle^3$	80	b_1	$\langle 1, p_1, p_2 \rangle^4$	91	76	15
CS6	$\langle 1, p_1, p_2, p_3 \rangle^4$	175	b_1	$\langle 1, p_1, p_2 \rangle^5 \cup \langle 1, p_1 \rangle^4 \langle p_3 \rangle$	196	170	26

specified in the design problem. Notice, that these equations are linear in the components of $\mathbf{B}$. We now assume that one component of $\mathbf{B}$ is eliminated using the constraint equation $\mathcal{C}_1$ in (15).

The number of monomials in the set $\langle 1, p_1, \dots, p_{3-p} \rangle^\mu$ is $M = \binom{\mu+3-p}{3-p}$. Multiplying the $m = n-1$ equations in a CSn problem by this list yields $Mm = (n-1)\binom{\mu+3-p}{3-p}$ equations $A\mathbf{y} = 0$. This expanded set contains monomials up to degree $\mu + 2$ in the components of of $\mathbf{p}$ together with monomials up to degree $\mu + 1$ in $\mathbf{p}$ multipled by each of the components of $\mathbf{B}$, which yields the monomial count

$$N = \binom{\mu+5-p}{3-p} + (2-b)\binom{\mu+4-p}{3-p}. \qquad (20)$$

If b_1 is used as the suppressed variable, then the identities can be obtained from the monomials in $p_1, \dots, p_{3-p}$ up to degree $\mu + 1$, or $L = \binom{\mu+4-p}{3-p}$.

The formulation of the eigenvalue elimination procedure for the nine design problems CSn, $3 \leq n \leq 6$, are summarized in Table 4. The column labeled "Multiplier" presents the linear product decomposition of the list of monomials that is used to multiply the original design equations to obtain the Mm equations $A\mathbf{y} = 0$. The suppresed variables are identified in the column "λ" and are used to formulate the identities. The column labeled "Identities" defines the monomial identities used to formulate the matrix. The last column gives the size of the final generalized eigenvalue problem, $N - r$, which in each case is equal to the LPD bound listed in Table 2.

The Case of CS4b. Now consider the case CS4b, which consists of $m = 3$ equations with one component each of $\mathbf{p}$ and $\mathbf{B}$ are specified, that is $p = b = 1$. This problem has the monomial structure

$$\mathcal{P}_i \in \langle 1, p_1, p_2\rangle\langle 1, p_1, p_2, b_1\rangle = \langle 1, p_1, p_2, b_1, p_1^2, p_1p_2, p_2^2, p_1b_1, p_2b_1\rangle,\ i = 1, 2, 3. \tag{21}$$

Notice that the variable b_2 is eliminated using the linear constraint $\mathcal{C}_1$ in (15). These equations are linear in b_1, so we use the monomials $\langle 1, p_1, p_2\rangle^\mu$ to expand this set of equations. Begin with $\mu = 0$ and increment μ to obtain:

- $\mu = 0$: This yields the multiplier list $\langle 1\rangle$ which yields the orignal three equations. The expanded equations take the form $A\mathbf{y} = 0$, where A is 3×9 matrix and has rank $r = 3$. However, $N - r = 9 - 3 = 6$ is less than the expected number of roots, given by LPD=7.
- $\mu = 1$: This yields the multiplier list $\langle 1, p_1, p_2\rangle$ which yields $A\mathbf{y} = 0$, such that A is 9×16. In this case, $16 - 9 = 7$, therefore we seek a set of seven identities. If we select b_1 as the suppresed variable, we have only the six monomials $\langle 1, p_1, p_2\rangle^2$ that we can use to construct identities. Considering p_1 instead, we hav an identity for each of the original nine monomials. However, $p_1\mathcal{P}_i = 0, i = 1, 2, 3$ are already in the set of equations $A\mathbf{y} = 0$, which means only six of the nine identities are independent. We obtain the same result if p_2 is chosen as the suppressed variable.
- $\mu = 2$: In this case, the multiplier list is $\langle 1, p_1, p_2\rangle^2$, and the equations become $A\mathbf{y} = 0$ where A has the dimensions 18×25. Now $25 - 18 = 7$, so we choose b_1 as the suppresed variable and generate identities using seven of the 10 elements in the list $\langle 1, p_1, p_2\rangle^3$. The result is a system $E(b_1)\mathbf{y} = 0$ that has full rank.

It useful to note that it is convenient to generate all of the identities associated with a specific suppressed variable and degree μ, and then simply check whether the system has full column rank.

The Case CS6. The same process can be applied to CS6 which has $m = 5$ design equations and $p = b = 0$. Applying the above formulas, we have the following table:

μ	N	Mm	r	$N - r$	L
0	18	5	5	13	4
1	40	20	20	20	10
2	75	50	50	25	20
3	126	100	100	26	35
4	196	175	170	26	56

The fact that the number of expanded equations $Mn = r$ for $\mu \leq 3$ is easily determined by numerical check. For $\mu \leq 2$, there are not enough identities to achieve full rank, because $L < N - r$. On the other hand for $\mu = 3$, while there seem to be plenty of identities, it fails the rank test. Thus, $\mu = 4$ results in a successful elimination procedure.

The reason that $\mu = 3$ fails to result in a successful elimination can be found in the structure of the design equations which can be written in the form

$$\mathcal{P}_i = \alpha_i + \beta_i b_2, \quad i = 1, \ldots, 5. \tag{22}$$

Note that α_i is of the form $\langle 1, p_1, p_2, p_3 \rangle \langle 1, p_1, p_2, p_3, b_1 \rangle$ and β_i is of the form $\langle 1, p_1, p_2, p_3 \rangle$.

Now, for any three $\ell, m, n \in \{1, 2, 3, 4, 5\}$, we must have the identity

$$D_{\ell mn} = \det \begin{pmatrix} \alpha_\ell & \beta_\ell & \mathcal{P}_\ell \\ \alpha_m & \beta_m & \mathcal{P}_m \\ \alpha_n & \beta_n & \mathcal{P}_n \end{pmatrix} = 0, \tag{23}$$

because the last column is a linear combination of the first two. All terms of this determinant take the form $\langle 1, p_1, p_2, p_3 \rangle^3 \mathcal{P}_i$ or $b_1 \langle 1, p_1, p_2, p_3 \rangle^2 \mathcal{P}_i$. Terms of the first type are linear combinations of monomials multiplier list. Terms of the second type are linear combinations of the multipler list and identities obtained for b_1. Thus, each $D_{\ell mn}$ is linearly dependent upon the expanded equations and identities. There are $\binom{5}{3} = 10$ such relations, so the 135×126 matrix in Eq.(19) has rank of only $135 - 10 = 125$, and the system cannot have full rank.

The case $\mu = 4$ is interesting because $r = 170 < Mm = 175$. The reason for this can again be seen in the structure of the design equations, now written in the form

$$\mathcal{P}_i = \alpha_i + \beta_i b_1 + \gamma_i b_2, \tag{24}$$

where α_i is of the form $\langle 1, p_1, p_2, p_3 \rangle^2$ and β_i, γ_i are both of the form $\langle 1, p_1, p_2, p_3 \rangle$. We now form the five identities

$$D_i = \det[\alpha_j \;\; \beta_j \;\; \gamma_j \;\; \mathcal{P}_j, j \neq i = 1, \ldots, 5] = 0, \quad i = 1, \ldots, 5. \tag{25}$$

The terms in these determinants are all of the form $\langle 1, p_1, p_2, p_3 \rangle^4 \mathcal{P}_i$, which linear combinations of the expanded set of equations. Thus, the rank of the expanded set of equations is $175 - 5 = 170$. Nielsen and Roth (1995) provide a similar analysis for this design problem.

7 Solution by Continuation

For polynomial systems with a large number of roots, elimination is not attractive, but we may find all solutions using polynomial continuation. For cases with $n = 7, 8$ task positions, the LPD bounds listed in Table 3 are large, so we attack these with continuation. We do not know at the outset whether the LPD bounds are sharp. By solving a generic example of each case, we can determine the exact root count for each problem. If it were to happen that the count is small, one could then be encouraged to look for an elimination method.

As it turns out, the LPD bounds are not sharp, but the number of roots is still too large to make an elimination approach desirable. Using PHC (Verchelde 1999), we computed the roots for random test cases. For each task position, we used a random number generator to obtain 7 numbers. Three are used as the position vector and the other 4 are normalized to a unit quaternion representing spatial orientation. With probability one, such a set of tasks will be generic; that is, the number of solutions to the synthesis problem defined by the tasks will be the generic root count. PHC includes an option to use a "random linear" start system, which will give exactly the number of continuation paths as the LPD bound. Some paths diverge to infinity, leaving a reduced number of finite roots. The root counts and execution times (1.5GHz Pentium 4) for the runs are summarized in Table 5. It should be noted that the solutions for these generic test problems can be used as start points in a parameter continuation (Morgan and Sommese, 1989). The number of continuation paths will then be reduced to actual root counts, hence the approximate running time will reduce by a factor of PHC/LPD, where these mean the root counts shown in Table 5.

Table 5: Root counts from PHC for $n = 7, 8$.

Name	CS7a	CS7b	CS7c	CS8
LPD	312	660	900	2184
PHC	186	216	588	804
time	0h15m	0h50m	1h23m	4h57m

8 Java Implementation and Numerical Examples for CS6

The generalized eigenvalue solution for the case CS6 has been implemented using pure Java language. The code has been integrated into our synthesis software SYNTHETICA (Su et.al. 2002) that allows the designer to specify the spatial task and then view and evaluate the resulting serial chains. The software is available online at *http://synthetica.eng.uci.edu/∼mccarthy/*. Testing a large number of sample problems shows that the average running time is 40ms on a 1.5GHz Pentium 4 system.

T_i	Long.	Lat.	Roll	x	y	z
1	0.00°	0.00°	0.00°	0.0000	0.0000	0.0000
2	23.20°	64.47°	-81.95°	-0.3627	-0.1324	0.3325
3	-50.36°	18.13°	-25.06°	-1.3539	-0.1925	1.4398
4	152.65°	8.87°	-30.05°	0.6485	-0.1308	1.0832
5	76.54°	40.79°	171.09°	-0.6574	-1.6225	1.4924
6	6.45°	5.30°	-8.07°	0.1769	1.2474	0.8503

Table 6: A set of six design positions that has 26 real solutions.

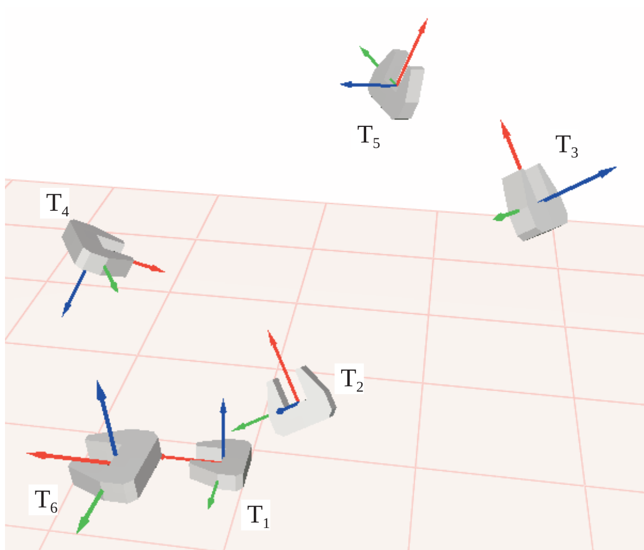

Figure 3: The set of six task positions in Table 6

We also wrote a Java program that generates five random task matrices (the other is fixed as identity matrix) with position limited in the box $|p_1| < 2.0, |p_2| < 2.0, 0.0 < p_3 < 2.0$ and orientation totally random. After solving about two million such task sets (took 20 hours), we found 11 examples that have 26 real solutions.

One of the problems with all real solutions is as follows. The six task positions are listed in the Table 6. The chosen vector $\mathbf{G} = (0.7831, -0.0723, -0.6176)$, and the random plane for defining $\mathbf{B}$ is $\mathbf{n} = (0.0879, -0.3730, 1), d = -0.5144$. The 26 real solutions computed by the eigenvalue method are listed in the Table 7. Figure 4 shows the 4th solution reaching the six design positions.

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Sol.	b_1	b_2	b_3	p_1	p_2	p_3
1	0.8156	0.8727	-0.2605	-1.3815	-0.2636	1.9431
2	2.0037	-0.9764	-1.0549	0.0621	-2.0535	1.1678
3	2.0542	-1.3361	-1.1935	13.1695	6.6146	6.1159
4	2.4830	-0.7165	-1.0000	-0.0979	-0.7653	-0.2029
5	2.4838	-3.3402	-1.9789	12.7054	-8.2452	4.2742
6	2.5155	-5.4135	-2.7551	3.7900	-3.4716	-6.5247
7	2.6960	1.4774	-0.2003	-11.5314	7.4024	-2.9912
8	2.7625	-1.3573	-1.2636	1.2585	-1.1873	-1.5185
9	2.9783	-1.4512	-1.3177	0.1603	-1.3765	-0.4616
10	3.0749	-2.3461	-1.6600	1.9543	-1.8996	-2.5701
11	3.1532	2.4199	0.1111	-1.0486	0.8682	4.8628
12	-3.6638	-3.0465	-1.3288	2.0468	1.7818	-9.3171
13	3.7042	-1.2844	-1.3193	1.7046	-1.0000	-1.2753
14	3.9600	-0.6776	-1.1154	0.9110	-1.8948	2.4928
15	4.5768	-2.4599	-1.8345	2.805	-1.7922	-2.4145
16	4.7375	-0.5066	-1.1200	0.5371	-0.1493	0.4314
17	4.8660	-1.1053	-1.3546	1.9477	-9.5564	-10.1024
18	5.0184	-1.8024	-1.6281	5.9946	2.9515	-3.9823
19	5.9291	-1.6356	-1.6459	1.0035	-1.0127	0.1820
20	6.4672	-4.6470	-2.8167	5.3990	-2.6060	-4.4818
21	6.5754	-6.6294	-3.5657	8.5861	-1.5185	-0.5701
22	-12.5378	18.2489	7.3959	15.2253	4.6261	12.795
23	14.0722	2.1352	-0.9552	-3.4604	7.2038	2.2306
24	18.7135	2.2086	-1.3359	4.9241	-5.9532	-5.8661
25	-21.0900	-7.1920	-1.3430	8.7098	-15.9104	-2.7448
26	-84.5932	112.1800	48.7732	73.1091	27.374	64.3705

Table 7: The 26 Real Solutions

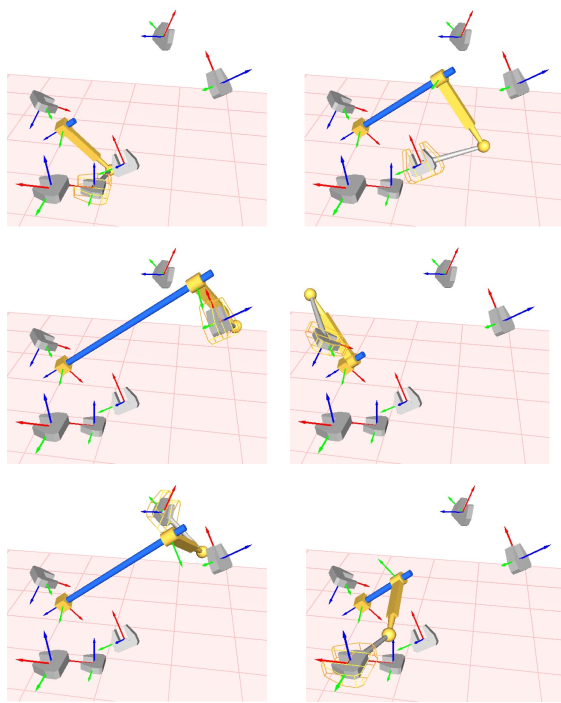

Figure 4: The 4th solution in the Table 7 is reaching the six design positions.

9 Conclusions

This paper examines the design of a cylindric PRS serial chain to reach a given set of spatial positions. A maximum of eight such task positions can be prescribed; we formulated nine different design problems with the number of task positions ranging from three to eight. All of these were analyzed using a linear product decomposition technique in order to determine a tighter bound the number of solutions than the total degree. Solutions based on reduction to generalized eigenvalue problems are provided for a variety of three, four, five, and six position design problems. Polynomial homotopy continuation was used to numerically determine the of solutions for the seven and eight position problems. In particular, the eight position synthesis problem has a total degree of 32768 with a linear product decomposition bound of 2184, while numerical experimentation yielded 804 solutions.

In practical design work, one would probably sacrifice one or more task positions to choose directly some of the linkage parameters.

A Java implementation of the generalized eigenvalue solution for the six task position problem has been integrated into our computer aided design tool SYNTHETICA to allow the designers to specify the spatial task and then view and evaluate the resulting serial chains.

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A Reduction of the Eigenvalue Problem to Size $M - \hat{E}$

We wish to reduce the sparse $M \times M$ generalized eigenvalue problem of Eq.19 to size $M - \hat{E}$, being the number of rows (and columns) in which the eigenvariable x appears. First, by re-ordering the monomials in $\mathbf{m}$ and the identity equations, we can always re-write the problem in block matrix form as

$$\begin{pmatrix} \hat{A}_1 & \hat{A}_2 & \hat{A}_3 & \hat{A}_4 \\ I_1 x + C_1 & C_2 & 0 & 0 \\ 0 & I_2 x & -I_2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \\ \mathbf{m}_4 \end{pmatrix} = 0, \tag{26}$$

where I_1, I_2 are identity matrices. In some cases, the last blockwise column is not present, but if it is, it must be full column rank. Using sparse Gaussian elimination, $\hat{A}_4$ can be reduced to upper triangular form yielding

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & U \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & 0 \\ I_1 x + C_1 & C_2 & 0 & 0 \\ 0 & I_2 x & -I_2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \\ \mathbf{m}_4 \end{pmatrix} = 0, \tag{27}$$

for some upper triangular matrix U. Pre-multiplying by the $(M - \hat{E}) \times M$ matrix

$$\begin{pmatrix} 0 & 0 & I_1 & 0 \\ 0 & I_2 & 0 & \tilde{A}_{23} \end{pmatrix},$$

gives the equation

$$\begin{pmatrix} I_1 x + C_1 & C_2 \\ \tilde{A}_{21} & \tilde{A}_{22} + \tilde{A}_{23} x \end{pmatrix} \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix} = 0, \tag{28}$$

where the trailing blocks have been dropped, since they are zero. The only computation involved is the triangularization of $\hat{A}_4$, which can be done efficiently by sparse routines. Eq.28 is the square generalized eigenvalue problem we seek.