# The Synthesis of an RPS Serial Chain to Reach a Given Set of Task Positions


Hai-Jun Su
suh@eng.uci.edu,

J. Michael McCarthy
jmmccart@uci.edu,

Robotics and Automation Laboratory, UCI


July 9, 2003


**Abstract**

*This paper examines the synthesis of the five degree-of-freedom robot formed by links connected by a revolute, prismatic and spherical joint to form an RPS serial chain. The reachable workspace of this robot is a right circular hyperboloid that defines a constraint equation with 10 dimensional parameters. Evaluating this equation on 10 arbitrary spatial positions yields a polynomial system of total degree 262,144. Polynomial continuation yields as many as 1020 RPS chains for a general 10 position task.*

*The number of task positions and constraint equations can be reduced by specifying some of the dimensional parameters. For the cases of six through eight task positions, analytical solutions are provided using both a resultant elimination strategy as well as a generalized eigenvalue elimination procedure. The generalized eigenvalue method has been implemented using Java and integrated into our design software. A numerical example of the solution for eight design positions is presented.*


# 1 Introduction

This paper examines the geometric design of a five degree-of-freedom robot constructed having a revolute (R), or hinged, joint at its the base, connected to a prismatic (P), or sliding joint, that supports a spherical wrist (S) and end-effector. This RPS serial chain has the property that the center of the spherical wrist (S) is constrained to move on a right circular hyperboloid around the axis of the revolute (R) joint. This constraint equation has 10 free parameters, therefore we can evaluate it on as many as 10 task positions. The result is design equations that we solve to define RPS chains that can reach the given set of task positions.

The design equations for the RPS chain were presented originally by Chen and Roth (1967) [1]. They considered the design of systems for which the direction of the revolute joint is specified, as well as the angle ($\alpha$) between this axis and the direction of prismatic joint (P). Because three of the ten parameters were specified, they considered RPS designs for a seven position task, and showed that there are at most 42 solutions. Nielsen and Roth (1995) [7] presented a generalized eigenvalue elimination strategy that yielded an analytical solution to this seven position deign problem. Kim and Tsai (2002) [2] considered the particular case when the direction of the prismatic joint is perpendicular to the axis of the revolute joint ($\alpha = 90°$), in which case the hyperboloid degenerates to a plane. They obtained 10 solutions for a general six position task. Su et al. (2003)[8] study the synthesis of the CS chain, which is the special case when the revolute and prismatic joints are parallel ($\alpha = 0°$). See McCarthy (2000)[5] for a general discussion of the design of planar, spherical and spatial open chains.

In this paper, we use the resultant based and matrix eigenvalue elimination strategy to obtain analytical solutions for RPS chains that reach six, seven and eight task positions. The nine and ten position cases are solved numerically using polynomial continuation (Wampler et al. 1990 [12], Lee and Mavroidis 2001 [3]).

The synthesis problems solved here can be used to design a variety of robotic systems. These can range from a single RPS chain which is actuated as an open-chain, five-degree-of-freedom robot, to a one-degree-of-freedom 5-RPS spatial mechanism having five RPS legs in parallel. Three RPS legs were used by Kim and Tsai (2002) [2] to design the three degree-of-freedom 3-RPS parallel manipulator.

# 2 The RPS Serial Chain

The RPS chain is defined by lines along the R and P joints and a point $\mathbf{P}$ at the center of the spherical joint, Figure 1 (a). Let $\mathbf{G} = (g_1, g_2, g_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$ be the direction of and a point on the R-joint axis respectively, $\mathsf{G} = (\mathbf{G}, \mathbf{B} \times \mathbf{G})$ be the Plucker coordinates of the axis of the R-joint, $\mathsf{H}$ be the line through point $\mathbf{P}$ parallel to the axis of the P-joint. Rotating the line $\mathsf{H}$ around $\mathsf{G}$ generates a right hyperboloid. We now denote as $\alpha$ and $a$ the twist angle around and distance along the common normal $\mathsf{N}$ between $\mathsf{G}$ and $\mathsf{H}$. For convenience, we choose $\mathbf{B}$ as the point where $\mathsf{G}$ and $\mathsf{N}$ intersect, or the center

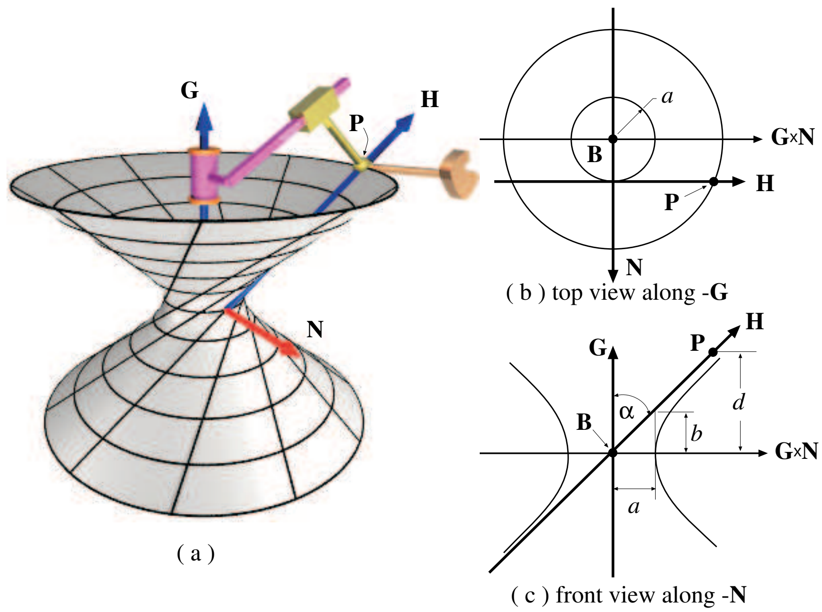


Figure 1: The RPS serial chain

of the hyperboloid. Figure 1 (b) and (c) are obtained by viewing (a) along negative N and negative G respectively.

From Figure 1 (b) and (c), we can compute the magnitude of $\mathbf{P}-\mathbf{B}$ as

$$(\mathbf{P}-\mathbf{B})^2 = a^2 + d^2 + (d\tan\alpha)^2, \tag{1}$$

where

$$d = \frac{(\mathbf{P}-\mathbf{B})\cdot\mathbf{G}}{\sqrt{\mathbf{G}\cdot\mathbf{G}}}, \tag{2}$$

is the component of $\mathbf{P}-\mathbf{B}$ in the direction $\mathbf{G}$. Notice that $\mathbf{G}$ is not required to be a unit vector.

Substitute (2) into (1), the constraint equation of the RPS chain becomes

$$(\mathbf{P}-\mathbf{B})^2 - ((\mathbf{P}-\mathbf{B})\cdot\mathbf{G})^2(\frac{1+\tan^2\alpha}{\mathbf{G}\cdot\mathbf{G}}) - a^2 = 0. \tag{3}$$

When $\alpha = 0$, the constraint equation becomes that of CS serial chain which is studied in Su et al. (2003)[8].

Expanding the constraint equation, we can write it in the form

$$q_0\mathbf{P}\cdot\mathbf{P} + \mathbf{Q}\cdot\mathbf{P} - (\mathbf{P}\cdot\mathbf{G})^2 - \zeta = 0, \tag{4}$$

where we introduce the parameters $q_0$, $\mathbf{Q} = (q_1, q_2, q_3)$ and $\zeta$ defined by

$$\begin{aligned}
q_0 &= \frac{\mathbf{G}\cdot\mathbf{G}}{1+\tan^2\alpha},\\
\mathbf{Q} &= 2(\mathbf{B}\cdot\mathbf{G})\mathbf{G} - 2q_0\mathbf{B}, \text{and}\\
\zeta &= (\mathbf{B}\cdot\mathbf{G})^2 - q_0\mathbf{B}\cdot\mathbf{B} + q_0 a^2.
\end{aligned} \tag{5}$$

Given values for $\zeta$, $q_0$, $\mathbf{Q}$, and $\mathbf{G}$, we can compute the coordinate vector $\mathbf{B}$ by solving the linear equations

$$\begin{Bmatrix} q_1\\ q_2\\ q_3 \end{Bmatrix} = 2\begin{bmatrix} g_1^2 - q_0 & g_1g_2 & g_1g_3\\ g_1g_2 & g_2^2 - q_0 & g_2g_3\\ g_1g_3 & g_2g_3 & g_3^2 - q_0 \end{bmatrix} \begin{Bmatrix} b_1\\ b_2\\ b_3 \end{Bmatrix} \tag{6}$$

Then the length and twist parameters, $a$ and $\alpha$, are obtained from the formulas

$$\alpha = \arccos\left(\sqrt{\frac{q_0}{\mathbf{G}\cdot\mathbf{G}}}\right), a = \sqrt{\frac{\zeta - (\mathbf{B}\cdot\mathbf{G})^2 + q_0\mathbf{B}\cdot\mathbf{B}}{q_0}}. \tag{7}$$

Thus, the 10 dimensional parameters $\zeta$, $q_0$, $\mathbf{Q}$, $\mathbf{G}$, and $\mathbf{P}$ define an RPS chain—recall that the length of $\mathbf{G}$ is fixed so it has only two independent parameters. In order to design an RPS serial chain, we evaluate the design equation (4) for up to 10 task positions.

# 3 Design Equations

In order to design a general RPS chain, we assume that we are given $n \leq 10$ goal positions defined by the transformations $[T_i], i = 1, \ldots, n$. We then seek a point $\mathbf{p} = (p_1, p_2, p_3)$ in the moving body that $\mathbf{P}_i = [T_i]\mathbf{p}$ satisfies the constraint equation (4) in each of the goal positions. The general solution for $n$ goal positions yields the following system of polynomials:

$$q_0\mathbf{P}_i\cdot\mathbf{P}_i + \mathbf{Q}\cdot\mathbf{P}_i - (\mathbf{P}_i\cdot\mathbf{G})^2 = \zeta, \quad 1 \leq i \leq n. \tag{8}$$

Subtract the first equation from the remaining equations to eliminate $\zeta$. This yields

$$\mathcal{P}_i: \quad q_0(\mathbf{P}_i\cdot\mathbf{P}_i - \mathbf{P}_1\cdot\mathbf{P}_1) + \mathbf{Q}\cdot(\mathbf{P}_i - \mathbf{P}_1) - (\mathbf{P}_i\cdot\mathbf{G})^2 + (\mathbf{P}_1\cdot\mathbf{G})^2 = 0, \quad 2 \leq i \leq n. \tag{9}$$

It is useful to note that the term $(\mathbf{P}_i\cdot\mathbf{P}_i - \mathbf{P}_1\cdot\mathbf{P}_1)$ is linear in $p_1, p_2, p_3$, because the quadratic terms cancel. The length of $\mathbf{G}$ is defined by the linear constraint

$$\mathcal{C}: \quad \mathbf{m}\cdot\mathbf{G} = d. \tag{10}$$

where $\mathbf{m}$ is an arbitrary vector and $d$ is an arbitrary constant. In the following sections, we seek solutions to the design equations (9) and (10) for 6 through 10 specified task positions, which we denote as design problems RPS$n, n = 6, \ldots, 10$.

We use the linear constraint (10) to eliminate $g_3$ in the set of polynomials $\mathcal{P}_i$, so the design parameters are $\mathbf{r} = (g_1, g_2, p_1, p_2, p_3, q_0, q_1, q_2, q_3)$. The case RPS10 yields in nine fourth degree polynomials $\mathcal{P}_i$ in these nine parameters. The total degree of this system of polynomials is $4^9 = 262,144$ which is known as the "Bezout bound" on the number of solutions.

A better bound on the number of solutions can be obtained by considering the monomial structure of the design equations. Let the notation $\langle 1, x_1, \ldots, x_n\rangle$ represent the set of linear combinations of the elements $1, x_1, \ldots, x_n$ with scalar coefficients, the the polynomials $\mathcal{P}_i$ have the structure

$$\mathcal{P}_i \in \langle 1, g_1, g_2\rangle^2 \langle 1, p_1, p_2, p_3\rangle \langle 1, p_1, p_2, p_3, q_0, q_1, q_2, q_3\rangle. \tag{11}$$

Polynomials in this set can be written as

$$\begin{aligned}\mathcal{P}_i = (\alpha_0 + \alpha_1 g_1 + \alpha_2 g_2)(\beta_0 + \beta_1 g_1 + \beta_2 g_2)(\gamma_0 + \gamma_1 p_1 + \gamma_2 p_2 + \gamma_3 p_3)\\ \times (\delta_0 + \delta_1 p_1 + \delta_2 p_2 + \delta_3 p_3 + \delta_4 q_0 + \delta_5 q_1 + \delta_6 q_2 + \delta_7 q_3)|_i = 0,\\ i = 2, \ldots, n.\end{aligned} \tag{12}$$

where the coefficients $\alpha_j$, $\beta_j$, $\gamma_j$ and $\delta_j$ are constants. This is known as the *linear product decomposition* of the design equations (Verschelde and Haegemans 1993 [10], Su et al. 2003 [8]).

# 4 Counting Solutions

The linear product decomposition of the design equations provides a convenient way to determine the effect of selecting values for the various design parameters. Verschelde and Haegemans (1993)[10] show that a bound on the number of roots to the design equations can be estimated from the set of equations (12). This is done by counting the admissible sets of linear terms, one taken from each equation. An admissible sets is defined by the fact that equating the terms to zero allows solution for all of the design parameters. This forces the associated terms in each equation to be zero and yields a root of the polynomial system.

The number of admissible sets from the linear decomposition (12) is obtained as follows. Let $g$, $p$ and $q$ denote the number of unknown design parameters among $(g_1, g_2)$, $(p_1, p_2, p_3)$ and $(q_0, q_1, q_2, q_3)$, respectively—notice that $g + p + q = n - 1$, which is the number of design equations $\mathcal{P}_i$ in (9). For example, the general case of RPS10 has $g = 2$, $p = 3$, $q = 4$.

First, we take $g$ terms of the form $\langle 1, g_1, g_2\rangle$ from the $g+p+q$ design equations (12). Since the degree of the set $\langle 1, g_1, g_2\rangle$ in (12) is two, we have two choices, which means the number of combinations is

$$2^g \binom{g+p+q}{g}. \tag{13}$$

Next, we can use $j=0$ up to $j=p$ terms of the form $\langle 1, p_1, p_2, p_3\rangle$ and combine them with $p+q-j$ terms taken from those of the form $\langle 1, p_1, p_2, p_3, q_0, q_1, q_2, q_3\rangle$ to define the $p+q$ unknown parameters among $(p_1, p_2, p_3)$ and $(q_0, q_1, q_2, q_3)$. This yields the combinations

$$\sum_{j=0}^{p} \binom{p+q}{j}. \tag{14}$$

Finally, combining these results, we obtain the linear decomposition bound for our design equations to be given by the formula

$$\mathrm{LPD}(g,p,q) = 2^g \binom{g+p+q}{g} \sum_{j=0}^{p} \binom{p+q}{j}. \tag{15}$$

This formula (15) can be used to determine a bound on the number of solutions for all variations of the design for RPS chains. See Table 1 where we have computed this bound for RPS10, as well as for the three versions of RPS9 that arise depending on which group of design variables contain the specified parameter. These bounds are too high to consider a variable elimination procedure for a solution.

| Name | RPS9a | RPS9b | RPS9c | RPS10 |
|---|---|---|---|---|
| $g$ | 1 | 2 | 2 | 2 |
| $p$ | 3 | 2 | 3 | 3 |
| $q$ | 4 | 4 | 3 | 4 |
| LPD | 1024 | 2464 | 4704 | 9216 |

Table 1: Linear product bounds for RPS10 and for the three versions of RPS9.

The design problem RPS8 allows us to specify two of the nine design parameters, in which case we obtain a significant simplification by defining the parameters $(g_1, g_2)$, because the $\mathcal{P}_i$ reduce to quadratic polynomials in the remaining parameters. The computation LPD(0,3,4)=64 is low enough that we seek a variable elimination procedure to find the roots for this design problem. Table 2 shows the LPD bounds for variations of the design problems RPS6 through RPS8. Notice that in each case we have specified the parameters $(g_1, g_2)$ to simplify the design equations. The analytic solutions of these cases show that the LPD bounds in Table 2 are exact.

| Name | RPS6a | RPS6b | RPS6c | RPS7a | RPS7b | RPS8 |
|---|---|---|---|---|---|---|
| $g$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $p$ | 1 | 2 | 3 | 2 | 3 | 3 |
| $q$ | 4 | 3 | 2 | 4 | 3 | 4 |
| LPD | 6 | 16 | 26 | 22 | 42 | 64 |

Table 2: Linear product bounds for variations of the design problems RPS6, PRS7, and RPS8.

# 5 Solution by Polynomial Continuation

Polynomial homotopy continuation smoothly transforms a as start system of polynomials with known roots into a desired target system, and tracks the path in order to determine all of the roots of the target system. The software PHC provided by Verschelde (1999) [11] provides a convenient way to obtain numerical solutions to our design equations.

In order to define a generic design tasks for RPS10 and RPS9a,b,c, we used a random number generator to obtain seven numbers for each task position. The first three numbers were used as the translation vector and the remaining four were normalized to define a unit quaternion defining a spatial orientation. The number of solutions to the design problems with these tasks will be the generic root count for the set of polynomials. The root counts and execution times on a 2.4GHz Pentium IV are shown in Table 3. These run times are for comparison only purposes only, it is possible to achieve much faster run-times by using parameter continuation with a known solution as the start system (Wampler et al. 1990[12]. In fact, using the mixed volume option on PHC the run-time can be reduced by a factor of 60.

It is interesting to notice that the actual number of roots far less than the total degrees of these polynomial systems, and an order of magnitude less than the LPD bounds.

# 6 Solution by Matrix Eigenvalue Elimination

For each of the design problems RPS6, RPS7 and RPS8 the LPD bounds are low enough to suggest that a variable elimination procedure may be convenient. Su et al (2003) [8] show how to use the monomial structure of the design equations for a CS chain and an array of matrix identities to obtain the elimination as a generalized matrix eigenvalue problem. Nielsen and Roth (1995) [7] applied a matrix eigenvalue elimination procedure to the problem RPS7b and obtain the 42 roots as solutions to a $42 \times 42$ generalized eigenvalue problem.

| Name | RPS9a | RPS9b | RPS9c | RPS10 |
|---|---|---|---|---|
| $n$ | 9 | 9 | 9 | 10 |
| $g$ | 1 | 2 | 2 | 2 |
| $p$ | 3 | 2 | 3 | 3 |
| $q$ | 4 | 4 | 3 | 4 |
| Deg | 65,536 | 65,536 | 65,536 | 262,144 |
| LPD | 1024 | 2464 | 4704 | 9216 |
| Roots | 280 | 232 | 792 | 1024 |
| Time | 1h15m | 3h45m | 8h8m | 23h34m |

Table 3: Linear product bound, number of roots, and PHC running time the generalized linear decomposition option.

In the following we apply this procedure to RPS8 to obtain a $64 \times 64$ generalized eigenvalue problem. And last, we briefly mention the matrix eigenvalue formulas for other cases.

## 6.1 Matrix Eigenvalue Elimination for RPS8

Recall that for RPS8 the vector $\mathbf{G}$ is specified by the designer, and the unknown parameters $\mathbf{r} = (p_1, p_2, p_3, q_0, q_1, q_2, q_3)$ are defined by the design equations with the structure,

$$\mathcal{P}_i \in \langle 1, p_1, p_2, p_3 \rangle \langle 1, p_1, p_2, p_3, q_0, q_1, q_2, q_3 \rangle|_i = 0, \quad i = 2, \dots, 8. \tag{16}$$

We reduce these equations to a univariate polynomial using generalize eigenvalue elimination by the following steps.

**Step 1.** Consider the parameters $(p_1, p_2, p_3)$. We form the list of monomials up to degree $\mu = 6$ in these three parameters, which has $M = \binom{3+\mu}{3} = 84$ entries, and then multiply the $m = 7$ equations (16) by this list to obtain $Mm = 588$ polynomials $\mathcal{Q}_j$ with the structure,

$$\mathcal{Q}_j \in \langle q_0, q_1, q_2, q_3 \rangle \langle 1, p_1, p_2, p_3 \rangle^{\mu+1} + \langle 1, p_1, p_2, p_3 \rangle^{\mu+2}. \tag{17}$$

The number $N$ of the monomials in each polynomial $\mathcal{Q}_j$ is computed to be

$$N = q\binom{\mu+1+3}{3} + \binom{\mu+2+3}{3} = 645. \tag{18}$$

**Step 2.** Let $\mathbf{y}$ be the vector formed by the $N$ monomials above, so the equations $\mathcal{Q}_j$ can be rewritten as $A\mathbf{y} = 0$, where $A$ is the $Mm \times N$, or $588 \times 645$, matrix of coefficients. Row reduce $A$ by Gaussian elimination to identify

$r$=rank[$A$] independent equations $B\mathbf{y} = 0$, where $B$ is an $r \times N$ matrix. The result is that $r = 581$, and we find that $N - r = 64$ is the number of roots predicted by the LPD bound.

**Step 3.** Now select one of the parameters $(q_0, q_1, q_2, q_3)$, say $q_0$, to be the suppressed variable. Let $M_k(p_1, p_2, p_3)$ be the set of degree $k$ monomials formed from the parameters $(p_1, p_2, p_3)$. We can use these monomials to construct the identities

$$\mathcal{R}_j : \quad z - q_0 x = 0, \quad j = 1, \ldots, L, \tag{19}$$

where

$$\begin{aligned} & x \in M_k(p_1, p_2, p_3), \\ \text{and} \quad & z \in (q_0) M_k(p_1, p_2, p_3), \quad k = 1, \ldots, \mu + 1 \end{aligned} \tag{20}$$

There are $L = \binom{\mu+1+3}{3} = 120$ of these identities.

**Step 4.** Form $N - r = 64$ of the identities (19) into the equation $[q_0 C - D]\mathbf{y} = 0$, where both $C$ and $D$ have entries that are simply 0 or 1, and assemble the $N \times N$ matrix equation,

$$E(q_0)\mathbf{y} = \begin{bmatrix} B \\ q_0 C - D \end{bmatrix} \mathbf{y} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{21}$$

In our case, a set of these identities can be found such that $E(q_0)$ has full rank for arbitrary values of $q_0$. This means that the $N - r = 64$ values of $q_0$ defined by $\det[E(q_0)] = 0$ are the desired solutions for our design equations. Su et al (2003) [8] show how to use the structure of the matrix $E(q_0)$ to reduce this to the solution of a generalized eigenvalue problem of dimension $N - r \times N - r$.

**Step 5.** The variables $\mathbf{p}$, $\mathbf{Q}$ can be obtained from the eigen vectors of the corresponding eigenvalue of $q_0$. And by substituting $q_0$, $\mathbf{p}$, $\mathbf{Q}$and $\mathbf{G}$ into the equation (6) and (7), we can obtain the values of the $a$, $\alpha$ and $\mathbf{B}$.

## 6.2 Other Cases

Using the similar steps in the case RPS8, we can easily obtain the matrix eigenvalue elimination formulas for RPS7 and RPS6. However here we only give the results for RPS7b.

For the case RPS7b, we have $m = 6$ design equations $\mathcal{P}_i$. Multiply design equations by monomials in ($p_1$, $p_2$, $p_3$) up to degree $\mu = 5$. The result is $r = 330$ independent equations in $N = 372$ monomials. With the addition of $N - r = 372 - 330 = 42$ monomial identities, we reduce this case to a $42 \times 42$ generalized eigenvalue problem (Table 4). Also see Nielsen and Roth (1995)[7] for a similar result.

| Case | $m$ | $q$ | $\mu$ | $M$ | $Mm$ | $r$ | $N$ | $L$ | $N-r$ |
|---|---|---|---|---|---|---|---|---|---|
| RPS8 | 7 | 4 | 6 | 84 | 588 | 581 | 645 | 120 | 64 |
| RPS7b | 6 | 3 | 5 | 56 | 336 | 330 | 372 | 84 | 42 |

Table 4: Eigenvalue elimination formulations for Tasks RPS8 and RPS7b

# 7 Solution by Resultant Elimination

The design equations for RPS, RPS7 and RPS6 in which **G** is specified ($g = 0$) can also be solved by a resultant elimination technique. For all these cases, we first eliminate the linear terms $(q_0, q_1, q_2, q_3)$ from the design equations (9) by taking the determinants of the all minors. Then we use Gaussian elimination to obtain additional equations to construct a square resultant matrix whose elements are polynomials of the suppressed variable. And last, instead of directly computing the determinant of this resultant matrix, we again convert it to a generalized eigenvalue problem. This will also improve numerical stability. After solving this polynomial, we back substitute to obtain the remaining design parameters.

Similarities in the structure of the different design problems allows us to collect these resultant formulations into three categories: (i) $p = 3$: RPS8, RPS7b, RPS6c, (ii) $p = 2$: RPS7a, RPS6b, and (iii) $p = 1$: RPS6a. We find that the procedure that generates the resultant for each category is essentially identical. Therefore we present only one case for each category in details and briefly mention the other cases.

## 7.1 Category (i): RPS8, RPS7b, RPS6c

The problems RPS8, RPS7b, RPS6c are related in that they have the same three quadratic parameters $(p_1, p_2, p_3)$. Recall that $n$ is the number of task positions, $m = n - 1 = p + q$ is the number $\mathcal{P}_i$ in the design equations (9) and $q = 4, 3, 2$ for these three cases respectively.

### 7.1.1 Resultant for RPS8

For the case RPS8, we have $n = 8, m = 7, p = 3, q = 4, g = 0$. See Table 5.

**Step 1.** The design equations (9) for RPS8 are linear in the parameters $(q_0, q_1, q_2, q_3)$, therefore we can rewrite these equations in the form,

$$\begin{bmatrix} \langle 1,p_1,p_2,p_3\rangle_{1,1} & \dots & \langle 1,p_1,p_2,p_3\rangle_{1,4} & \langle 1,p_1,p_2,p_3\rangle^2_{1,5} \\ \vdots & \vdots & \vdots & \vdots \\ \langle 1,p_1,p_2,p_3\rangle_{7,1} & \dots & \langle 1,p_1,p_2,p_3\rangle_{7,4} & \langle 1,p_1,p_2,p_3\rangle^2_{7,5} \end{bmatrix} \begin{Bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix}, \tag{22}$$

or

$$[D(p_1, p_2, p_3)]\mathbf{q} = 0. \tag{23}$$

This set of equations has a solution if all of the $5 \times 5$ minors of the $7 \times 5$ matrix $[D(p_1, p_2, p_3)]$ are zero. For convenience in the following we relabel $p_1 = x$, $p_2 = y$ and $p_3 = z$.

**Step 2.** The minors yield $\binom{m}{q+1} = \binom{7}{5} = 21$ polynomials $\mathcal{Q}_i$ of degree $q+2 = 6$ in the parameters $(x, y, z)$. Now if we suppress the variable $z$, there are total $\binom{n}{p-1} = \binom{8}{2} = 28$ monomials in $x$ and $y$ in $\mathcal{Q}_i$ . Clearly, to construct a square $28 \times 28$ resultant matrix in $z$, we have to find $28 - 21 = 7$ extra equations without introducing new terms in $x$ and $y$. The process is as follows: Collect terms in these equations $\mathcal{Q}_i$ by factoring out the 13 sixth degree monomials that are of zero and first degree in $z$, that is

$$\mathbf{v} = (x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, x^5z, x^4yz, x^3y^2z, x^2y^3z, xy^4z, y^5z, 1)^T. \tag{24}$$

This allows us to write the 21 polynomials as the matrix equation

$$[A_{21\times 14}]\mathbf{v} = 0. \tag{25}$$

The first 13 columns of $C$ are constants and the last column is a linear combination of the remaining monomials.

**Step 3.** The matrix $A$ is now row reduced to define $B$ such that the last eight rows are polynomials with no sixth degree monomials that are of degree zero and one in $z$. From these last eight polynomials we construct

$$\mathcal{R}_j : x\mathcal{Q}_{13+j} = 0 \quad \text{and} \quad \mathcal{S}_j : y\mathcal{Q}_{13+j} = 0, \quad j = 1, \dots, 8. \tag{26}$$

Three pairs, denoted $\mathcal{R}_k$ and $\mathcal{S}_k$, $k = 1, 2, 3$, can be selected from these two sets such that together with the $\mathcal{Q}_i$, $i = 1, \dots, 21$, we have 27 linearly independent polynomials. This is easily determined by checking the rank of the combined coefficient matrix for random values of $z$.

**Step 4.** Select one of the six polynomials $\mathcal{R}_k$ and $\mathcal{S}_k$ and eliminate the seven sixth degree terms in $x$ and $y$ by direct substitution using (25). Denote this polynomial as $\mathcal{R}'$, then a final polynomial is obtained as

$$\mathcal{T} : x\mathcal{R}' = 0. \tag{27}$$

The 28 polynomials $\mathcal{Q}_i, i = 1, \ldots, 21$, $\mathcal{R}_k$ and $\mathcal{S}_k$, $k = 1, 2, 3$, and $\mathcal{T}$ can now be organized for solution by a resultant.

**Step 5.** Suppress the parameter $z$ in the polynomials $\mathcal{Q}_i$, $\mathcal{R}_k$, $\mathcal{S}_k$ and $\mathcal{T}$ and collect coefficients of the 28 power products up to the sixth degree in $x$ and $y$,

$$\mathbf{w} = (x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, \ldots, x^2, xy, y^2, x, y, 1)^T. \tag{28}$$

The result is the matrix equation

$$[C(z)_{28\times 28}]\mathbf{w} = 0. \tag{29}$$

This equation has a solution only if the $\det[C(z)] = 0$ which is the resultant for this system of polynomials.

To verify the degree of this polynomial, we consider the the upper bound on the exponent of $z$ for each element of $C(z)$. See equation (30).

$$\begin{array}{r} \mathcal{Q}_i\left\{\begin{array}{l} \\ \\ \\ \end{array}\right. \\ \mathcal{R}_k\left\{\begin{array}{l} \\ \\ \\ \end{array}\right. \\ \mathcal{S}_k\left\{\begin{array}{l} \\ \\ \\ \end{array}\right. \\ \mathcal{T}\left\{ \right. \end{array}\left|\begin{array}{c|cccccccc} \overbrace{0\ldots0}^{m} & \overbrace{1\ldots1}^{m-1} & \ldots & \multicolumn{3}{c}{\overbrace{m-3 \quad m-3 \quad m-3}^{3}} & \multicolumn{2}{c}{\overbrace{m-2 \quad m-2}^{2}} & \overbrace{m-1}^{1} \\ \vdots & \vdots & \vdots & & \vdots & & & \vdots & \vdots \\ 0\ldots0 & 1\ldots1 & \ldots & m-3 & m-3 & m-3 & m-2 & m-2 & m-1 \\ \hline 1\ldots1 & 2\ldots2 & \ldots & m-2 & m-2 & m-2 & m-1 & m-1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & & & \vdots & \vdots \\ 1\ldots1 & 2\ldots2 & \ldots & m-2 & m-2 & m-2 & m-1 & m-1 & 0 \\ \hline 1\ldots1 & 2\ldots2 & \ldots & m-2 & m-2 & m-2 & m-1 & m-1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & & & \vdots & \vdots \\ 1\ldots1 & 2\ldots2 & \ldots & m-2 & m-2 & m-2 & m-1 & m-1 & 0 \\ \hline 2\ldots2 & 3\ldots3 & \ldots & m-1 & m-1 & m-1 & m-1 & m-1 & 0 \end{array}\right| \tag{30}$$

The upper bound on the degree of $\det[C(z)] = 0$ can be determined by summing up the diagonal exponents from lower left corner to upper right corner, that is

$$D_m^{(i)} = (2 + (m-1)) + \sum_{j=1}^{m-1} j(m-j) = \frac{(m^2 - m + 6)(m+1)}{6} \tag{31}$$

Substituting $m = 7$ into (31), we find that the RPS8 problem has up to 64 solutions, which is exactly predicted by LPD bound in Table 2.

| Name | RPS6a | RPS6b | RPS6c | RPS7a | RPS7b | RPS8 |
|---|---|---|---|---|---|---|
| $n$ | 6 | 6 | 6 | 7 | 7 | 8 |
| $m$ | 5 | 5 | 5 | 6 | 6 | 7 |
| $g$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $p$ | 1 | 2 | 3 | 2 | 3 | 3 |
| $q$ | 4 | 3 | 2 | 4 | 3 | 4 |
| $\binom{m}{q+1}$ | 1 | 5 | 10 | 6 | 15 | 21 |
| $\binom{n}{p-1}$ | 1 | 6 | 15 | 7 | 21 | 28 |
| $D_m$ | 6 | 16 | 26 | 22 | 42 | 64 |

Table 5: The resultant formulas for RPS6, PRS7, and RPS8.

### 7.1.2 RPS7b

We form the resultant for the design problem RPS7b by following the steps presented above for RPS8:

1. Assemble the six design equations $\mathcal{P}_i, i = 2, \ldots, 7$ as linear combinations of the three parameters $\mathbf{q} = (q_0, q_1, q_2)$,

$$[D(p_1, p_2, p_3)]\mathbf{q} = 0, \tag{32}$$

where $[D(p_1, p_2, p_3)]$ is a $6 \times 4$ matrix. Relabel the parameters $p_1 = x$, $p_2 = y$ and $p_3 = z$;

2. The $\binom{6}{4} = 15$ $4 \times 4$ minors yield the fifth degree polynomials $\mathcal{Q}_i, i = 1, \ldots, 15$. Collect the fifth degree monomials that are of degree zero and one in $z$ into the vector

$$\mathbf{v} = (x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, x^4z, x^3yz, x^2y^2z, xy^3z, y^4z, 1)^T, \tag{33}$$

so we have

$$[A_{15\times12}]\mathbf{v} = 0. \tag{34}$$

The first 11 columns of this matrix are constant while the last column is a linear combination of the remaining monomials.

3. Row reduce $A$ to obtain $B$ such that the last four rows consist of polynomials without the fifth degree monomials that have $z$ of degree zero or one. From these last four rows, we construct the pairs of polynomials

$$\mathcal{R}_j : x\mathcal{Q}_{11+j} = 0 \quad \text{and} \quad \mathcal{S}_j : y\mathcal{Q}_{11+j} = 0, \quad j = 1, \ldots, 4. \tag{35}$$

Select five of these polynomials $\mathcal{R}_k$ and $\mathcal{S}_k$, $k = 1, 2$ and $\mathcal{R}_3$, such that together with $\mathcal{Q}_i, i = 1, \ldots, 15$, the set is linearly independent.

4. Chose the polynomial $\mathcal{R}_1$ and eliminate the fifth degree terms using (34) to define $\mathcal{R}'$, then the last polynomial is $\mathcal{T} = x\mathcal{R}'$.

5. Suppress the parameters $z$ in the set of polynomials $\mathcal{Q}_i, i = 1, \ldots, 15$, $\mathcal{R}_k$ and $\mathcal{S}_k$, $k = 1, 2$ and $\mathcal{R}_3$ and $\mathcal{T}$ to obtain the resultant matrix $[C(z)_{21\times 21}]$. The determinant of this matrix yields a polynomial of degree 42, which yields the solutions to the RPS7b design equations. Note that this degree has been predicted in equation (31). See Table 5. Also see the LPD bound in Table 2.

#### 7.1.3 RPS6c

The five design equations for RPS6c have two linear parameters $(q_0, q_1)$ and we construct the ten $3 \times 3$ minors from these equations, in order to obtain the quartic polynomials $\mathcal{Q}_i, i = 1, \ldots, 10$. We construct an additional four polynomials $\mathcal{R}_k$ and $\mathcal{S}_k$, $k = 1, 2$ and then a fifth polynomial $\mathcal{T}$ following the process described above. We suppress the variable $z$ in these 15 polynomials to obtain the resultant matrix $[C(z)_{15\times 15}]$. The determinant of this resultant yields a polynomial of degree 26, which can also be computed from equation (31). The roots of this polynomial define the solutions to our design problem RPS6c.

### 7.2 Category (ii): RPS7a and RPS6b

The problems RPS6b and RPS7a are related in that they have the same two quadratic parameters $(p_1, p_2)$, and the difference in the number of design equations arises from the differing number of linear unknowns, $(q_0, q_1, q_2)$ and $(q_0, q_1, q_2, q_3)$, respectively. Because the challenge centers on the elimination of the quadratic parameters $(p_1, p_2)$, the process of obtaining the resultant is essentially the same for both problems. We develop this in details for RPS7a, and then outline the process for RPS6b.

#### 7.2.1 RPS7a

The six design equations for RPS7a have the form

$$\begin{bmatrix} \langle 1, p_1, p_2\rangle_{1,1} & \ldots & \langle 1, p_1, p_2\rangle_{1,4} & \langle 1, p_1, p_2\rangle^2_{1,5} \\ \vdots & \vdots & \vdots & \vdots \\ \langle 1, p_1, p_2\rangle_{6,1} & \ldots & \langle 1, p_1, p_2\rangle_{6,4} & \langle 1, p_1, p_2\rangle^2_{6,5} \end{bmatrix} \begin{Bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix}, \tag{36}$$

or

$$[D(p_1, p_2)]\mathbf{q} = 0. \tag{37}$$

This set of equations has a solution if all of the $5 \times 5$ minors the $6 \times 5$ matrix $[D(p_1, p_2, p_3)]$ are zero. For convenience in the following we relabel $p_1 = x$, and $p_2 = y$. These minors yield six polynomials $\mathcal{Q}_i, i = 1, \ldots, 6$ that are of degree six in $x$ and $y$.

Suppress the parameter $y$ and define the vector $\mathbf{v} = (x^6, x^5, x^4, x^3, x^2, x, 1)^T$ so the polynomials $\mathcal{Q}_i$ can be written in the form

$$A(y)\mathbf{v} = 0, \tag{38}$$

where $A(y)$ is a $6 \times 7$ matrix. The first column of $A(y)$ has constant elements.

We now row reduce $A(y)$ so the last five rows have no $x^6$ term. Select one of these five polynomials, denoted $\mathcal{R}$, and construct the polynomial $\mathcal{T} = x\mathcal{R}$. The seven polynomials $\mathcal{Q}_i, i = 1, \ldots, 6$ and $\mathcal{T}$ form the $7 \times 7$ matrix equation

$$C(y)\mathbf{v} = 0. \tag{39}$$

The matrix $C(y)$ is the required resultant matrix.

As in the category (i), we can also verify the degree of the final polynomial by considering the the upper bound on the exponent of $y$ for each element of $C(y)$ as follows,

$$\begin{array}{r} \mathcal{Q}_i \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. \\ \mathcal{T} \{ \end{array} \left| \begin{array}{ccccc} 0 & 1 & \cdots & m-1 & m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & m-1 & m \\ 1 & 2 & \cdots & m & m \end{array} \right| \tag{40}$$

The upper bound on the degree of $\det[C(y)] = 0$ can be determined by summing up the diagonal exponents from lower left corner to upper right corner, that is

$$D_m^{(ii)} = 1 + \sum_{j=1}^{m} j = 1 + \frac{m(m+1)}{2} \tag{41}$$

As a result, the determinant $\det[C(y)] = 0$ is a polynomial of degree 22 in $y$ which is solved to determine designs for RPS7a.

### 7.2.2 RPS6b

The five design equations for RPS6b can be written a form similar to (45) except $[D(p_1, p_2)]$ is now of dimension $5 \times 4$. In this case, a solution exists if the five $4 \times 4$ minors are zero. This yields five polynomials $\mathcal{Q}_i, i = 1, \ldots, 5$ of degree five. Relabel $p_1 = x$, and $p_2 = y$, and suppress $y$, so we have

$$A(y)\mathbf{v} = 0, \tag{42}$$

where $A(y)$ is a $5 \times 6$ matrix and $\mathbf{v} = (x^5, x^4, x^3, x^2, x, 1)^T$. The first column of $A(y)$ has constant elements and can be row reduced to eliminate the $x^5$ terms in the last four rows. Multiply one of these rows by $x$ to construct a sixth polynomial $\mathcal{T}$, and we obtain

$$C(y)\mathbf{v} = 0. \tag{43}$$

The $6 \times 6$ matrix $C(y)$ is the required resultant matrix, and its determinant yields a polynomial of degree 16 in $y$. This degree has been predicted in (41). We solve this to determine designs for RPS6b.

## 7.3 Category (iii): RPS6a

In this category, there only one quadratic variable. The five design equations for the design problem RPS6a have the form

$$\begin{bmatrix} \langle 1, p_1 \rangle_{1,1} & \dots & \langle 1, p_1 \rangle_{1,4} & \langle 1, p_1 \rangle_{1,5}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \langle 1, p_1 \rangle_{5,1} & \dots & \langle 1, p_1 \rangle_{5,4} & \langle 1, p_1 \rangle_{5,5}^2 \end{bmatrix} \begin{Bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix}, \tag{44}$$

or

$$[D(p_1)]\mathbf{q} = 0. \tag{45}$$

The $5 \times 5$ matrix $[D(p_1)]$ is the resultant for this system of polynomials. The determinant $\det[D(p_1)]$ is a polynomial of degree six which is solved to obtain the designs for RPS6a.

## 7.4 Other Cases

It is possible to explore $n < 6$ cases in which parameters $g_1$ and $g_2$ are specified. These are either trivial or can be solved using the techniques presented in the previous sections. We do not discuss them here. However, we have not obtained resultant formulations for the cases RPS6 and RPS7 in which the parameters $g_1$ and $g_2$ remain as unknowns. These are challenging problems and so far we have only polynomial continuation results.

## 7.5 Roots of Matrix Polynomials

One of the disadvantages of solving a set of polynomials using a resultant matrix is the two step process of computing the determinant and then finding the roots of the resulting polynomial. This process is numerically unstable when the degree of the polynomial is high. Fortunately, this can be simplified by reformulating the resultant as a generalized eigenvalue problem and finding the roots using eigenvalue solution techniques (Manocha and Krishnan, 1996 [4]) .

Each of our resultant matrices $[C(z)]$ can be expanded into an $m \times m$ matrix polynomials of degree $d$,

$$C(z) = C_0 + C_1 z + C_2 z^2 + \ldots + C_d z^d \tag{46}$$

where $C_i$ are matrices of constants. Our goal is to compute the roots of $\det[C(z)] = 0$. To do this, we construct the two matrices $A_1$ and $A_2$ that have $d$ rows and columns consisting of $m \times m$ matrices, given by

$$A_1 = \begin{bmatrix} I_m & 0 & 0 & \cdots & 0 \\ 0 & I_m & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & I_m & 0 \\ 0 & 0 & \cdots & 0 & C_d \end{bmatrix} \tag{47}$$

and

$$A_2 = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ -C_0 & -C_1 & -C_2 & \cdots & -C_{d-1} \end{bmatrix}, \tag{48}$$

where $I_m$ is the $m \times m$ identity matrix and 0 is an $m \times m$ matrix of zeros. These two matrices have dimension $dm \times dm$.

Manocha and Krishnan (1996) show that the solutions to $\det[C(z)] = 0$ are among the $dm$ eigenvalues obtained from

$$[\lambda A_1 - A_2] = \begin{bmatrix} \lambda I_m & -I_m & 0 & \cdots & 0 & 0 \\ 0 & \lambda I_m & -I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda I_m & -I_m \\ C_0 & C_1 & C_2 & \cdots & C_{d-2} & \lambda C_d + C_{d-1} \end{bmatrix}. \tag{49}$$

It is possible to compute $\det[\lambda A_1 - A_2]$ symbolically by using column operations to eliminate the $-I_m$. The determinant of the resulting block triangular matrix is easily evaluated and seen to equal $\det[C(z)]$ for $\lambda = z$. The finite eigenvalues obtained from (49) are the desired roots of our resultant polynomial.

## 7.6 Back Substitution

After obtaining the solutions for the suppressed variable, we substitute it back into original equations (9) to solve the remaining variables. Typically, we can solve the linear equations of the resultant matrix for the variables in $(p_1, p_2, p_3)$. The remaining variables can be determined using the original design equations (9). The physical dimension of the RPS chain, $a$, $\alpha$ and $\mathbf{B}$ can be obtained from equations (6) and (7), as we did in the matrix eigenvalue method.

| | $x$ | $y$ | $z$ | Long. | Lat. | Roll |
|---|---|---|---|---|---|---|
| $T_1$ | 0.0000 | 0.0000 | 0.0000 | 0.00° | 0.00° | 0.00° |
| $T_2$ | -0.7971 | -1.6281 | 0.8635 | -127.44° | 72.75° | -120.02° |
| $T_3$ | -0.7895 | 0.6463 | 0.0567 | 17.58° | -18.29° | 25.47° |
| $T_4$ | 0.0000 | 1.9789 | 1.5815 | 110.33° | -23.02° | -77.72° |
| $T_5$ | 1.8715 | 1.4191 | 0.5000 | -2.90° | 7.46° | -7.80° |
| $T_6$ | 1.2963 | 1.1081 | 1.6690 | 17.51° | 59.28° | 14.12° |
| $T_7$ | -0.4100 | -1.5349 | 1.5575 | 104.56° | 81.12° | 2.65° |
| $T_8$ | -0.2412 | -0.0496 | 1.2322 | -44.35° | -39.22° | -25.90° |

Table 6: Eight Task Positions

# 8 Implementation and Numerical Example

Both the resultant based and the eigenvalue elimination solutions were implemented using Mathematica. The results were essentially identical, and the notebooks are available from the authors on request.

We have implemented the matrix eigenvalue elimination procedure in Java, and integrated the algorithm into our synthesis software SYNTHETICA (Su et.al. 2002[9]). This allows a designer to specify eight task positions, and compute the resulting RPS chains. These chains can then be animated and evaluated. The software is available online at *http://synthetica.eng.uci.edu/∼mccarthy/*. Test problems show that the average solution time is 0.3 seconds on a 2.4GHz Pentium 4 system.

The eight random task positions were generated by using a set of three random numbers to define the translation vectors, and a normalized set of four random numbers for the quaternion that defines the rotation. The eight positions were then multiplied by the inverse of the first position in order to define the relative positions listed in the Table 8 and shown in Figure 2. The direction of R-joint axis was randomly chosen to be $\mathbf{G} = (-0.5051, -0.1055, 0.8566)$.

The computation yields 36 real solutions among a total of 64 roots, however only 19 of these have physical meaning. This is due to the requirement that $a^2 > 0$ and $b^2 > 0$ ($\tan^2\alpha > 0$). Recall $\tan\alpha = a/b$. See Figure 1. Table 7 presents the 36 solutions, of which the first 19 correspond to real RPS chains. Figure 3 that the solution no. 8 reaching each of the eight specified task positions.

# 9 Conclusions

This paper examines the design of an RPS serial chain that reach a prescribed set of spatial positions. The general case requires the solution of 10 quartic polynomials, one for each specified task position, in 10 unknown dimensional

| Sol. | $a^2$ | $b^2$ | $b_1$ | $b_2$ | $b_3$ | $p_1$ | $p_2$ | $p_3$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.405 | 0.203 | -0.982 | 8.719 | 0.958 | -2.136 | 2.799 | 4.125 |
| 2 | 0.478 | 0.043 | 0.313 | 0.076 | 0.406 | 0.490 | -1.932 | 0.921 |
| 3 | 0.953 | 0.261 | 0.878 | -0.879 | 0.225 | 0.188 | -2.012 | 0.166 |
| 4 | 1.589 | 0.108 | 0.026 | -1.727 | 1.693 | -1.316 | -0.974 | 0.576 |
| 5 | 1.679 | 1.682 | -0.426 | 1.831 | -2.176 | -1.148 | 0.905 | -2.723 |
| 6 | 1.896 | 0.595 | 1.353 | -3.223 | 0.072 | 0.057 | -3.166 | -0.496 |
| 7 | 2.265 | 0.421 | -3.036 | 1.718 | -2.071 | -1.763 | -0.700 | -2.714 |
| **8** | **2.328** | **1.073** | **-0.287** | **1.433** | **1.274** | **-0.781** | **-0.062** | **1.083** |
| 9 | 2.406 | 0.280 | -0.044 | 0.729 | 1.289 | -0.431 | -0.886 | 1.098 |
| 10 | 2.618 | 3.033 | -1.643 | 8.192 | 0.428 | -3.199 | 4.171 | 3.151 |
| 11 | 2.698 | 0.293 | -0.072 | 0.843 | 1.268 | -0.360 | -0.954 | 1.165 |
| 12 | 2.971 | 0.949 | -0.210 | 1.594 | 1.194 | -0.568 | -0.363 | 1.317 |
| 13 | 3.885 | 6.322 | -2.646 | 2.838 | 0.659 | -3.701 | 4.107 | -1.209 |
| 14 | 4.998 | 1.285 | -0.318 | 1.989 | 1.111 | -0.287 | -0.561 | 1.494 |
| 15 | 6.330 | 1.024 | -2.020 | 4.802 | -0.061 | 0.619 | 1.888 | -0.182 |
| 16 | 26.71 | 2.607 | -1.967 | 9.317 | -1.970 | 3.464 | 2.743 | -1.939 |
| 17 | 26.97 | 7.404 | -2.855 | -4.686 | 2.864 | -5.330 | 0.720 | -0.262 |
| 18 | 216.4 | 18.53 | -4.086 | 24.57 | -5.625 | 8.502 | 7.306 | -5.474 |
| 19 | 323.3 | 12.70 | 2.544 | -24.88 | -1.376 | -0.084 | -6.108 | 0.838 |
| 20 | -0.227 | -0.041 | 0.618 | -0.995 | 0.453 | -0.048 | -1.458 | -0.660 |
| 21 | -0.843 | -1.596 | -0.996 | 6.044 | 0.404 | -3.326 | 3.747 | 2.535 |
| 22 | -2.060 | -0.211 | -7.252 | -0.182 | -1.168 | -5.068 | 1.078 | -0.768 |
| 23 | -5.733 | -0.149 | -4.297 | 2.389 | -1.732 | -0.934 | -1.094 | -1.187 |
| 24 | 8.449 | -0.559 | -0.298 | 1.652 | 0.989 | 0.126 | -0.910 | 0.542 |
| 25 | 9.324 | -1.375 | -0.344 | 1.584 | 1.180 | 0.426 | -1.114 | 0.755 |
| 26 | 9.383 | -9.574 | -1.122 | 0.528 | 3.293 | 0.198 | -1.016 | 0.436 |
| 27 | 11.74 | -0.687 | -0.344 | 1.976 | 0.664 | 0.453 | -1.038 | 0.358 |
| 28 | 14.74 | -1.997 | -0.242 | 1.920 | 0.724 | 1.0767 | -1.496 | 0.553 |
| 29 | 27.08 | -21.33 | -1.977 | 1.745 | 4.644 | 0.188 | -0.908 | 0.820 |
| 30 | 38.97 | -1.829 | -0.551 | 4.229 | 1.596 | 1.146 | -0.487 | 1.061 |
| 31 | 39.09 | -1.713 | 2.729 | -2.716 | 2.497 | -1.467 | -0.684 | -0.460 |
| 32 | 66.70 | -69.62 | -0.687 | -0.898 | -0.328 | 4.335 | -3.955 | -6.188 |
| 33 | 75.45 | -6.349 | -0.993 | 5.338 | 0.313 | 3.087 | -1.277 | 0.534 |
| 34 | 138.7 | -431.3 | -9.865 | 1.988 | 15.83 | 1.959 | -1.637 | 1.902 |
| 35 | 188.3 | -166.6 | -5.968 | 5.979 | 7.336 | 3.893 | -1.804 | 2.664 |
| 36 | 7248. | -6889. | 1.031 | -2.554 | -0.296 | -58.89 | 36.30 | 43.72 |

Table 7: The 36 Real Solutions

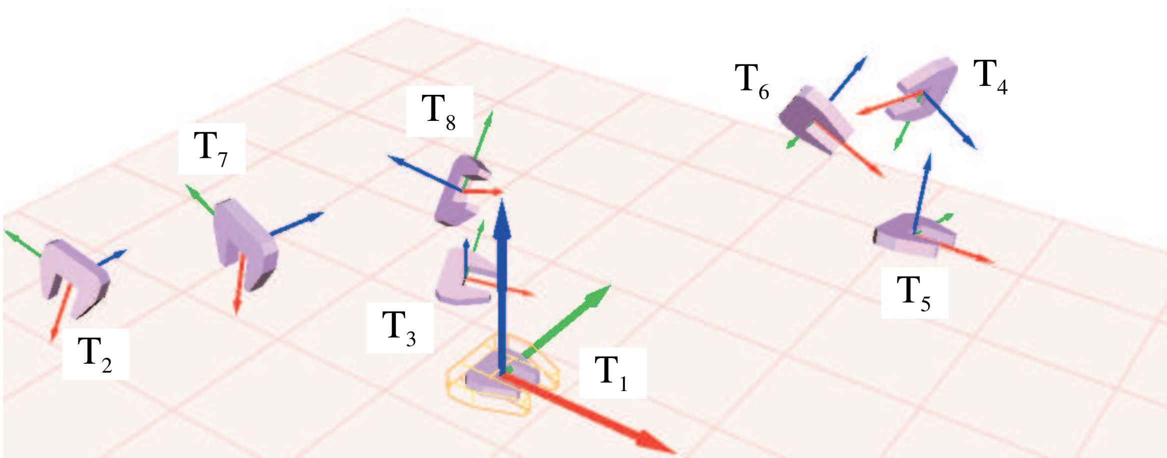


Figure 2: The eight positions shown in SYNTHETICA

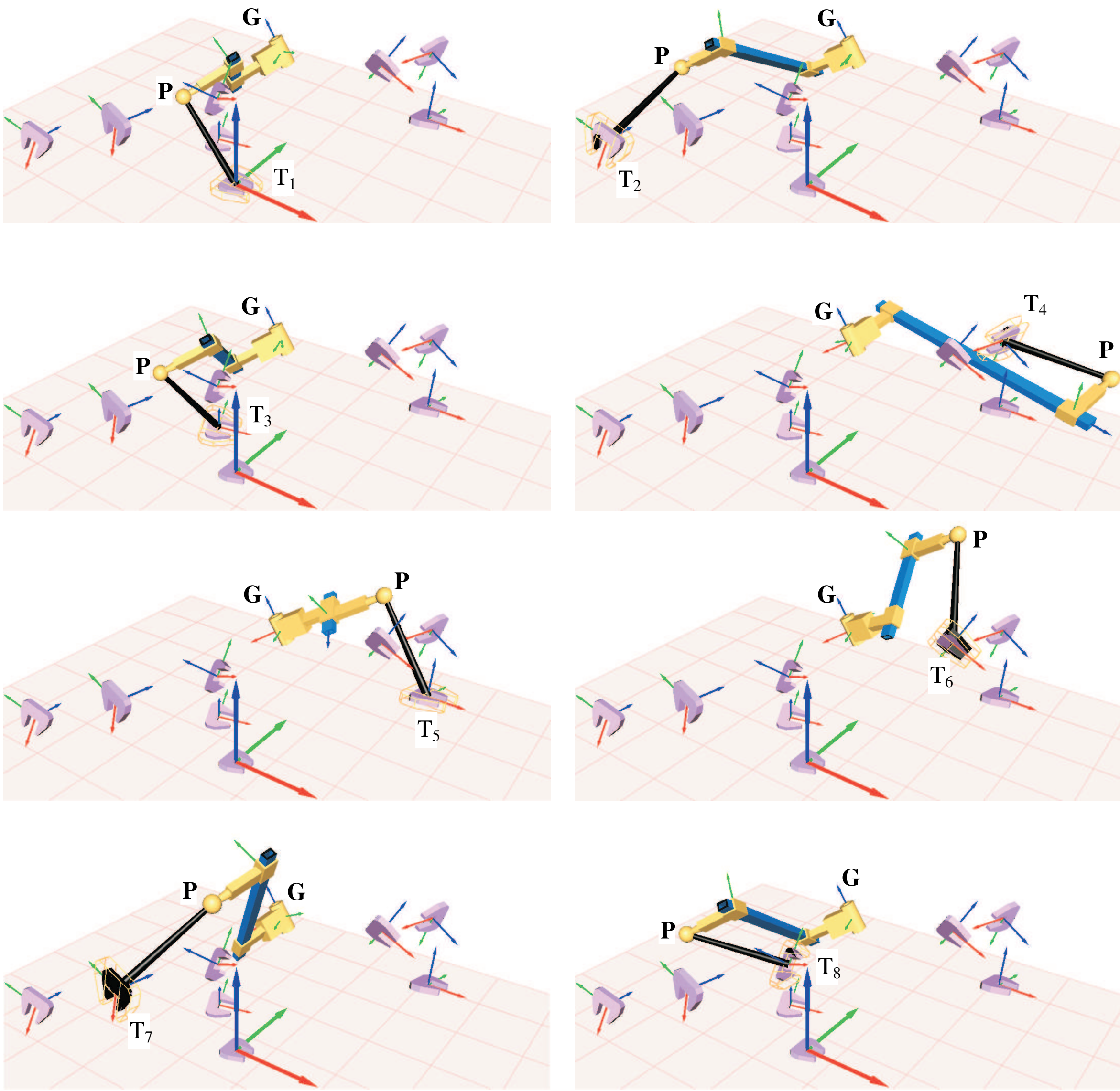


Figure 3: Solution 8 in Table 7 is shown reaching the eight task positions.

parameters. We solve this problem using polynomial continuation and find that the system can have as many as 1020 roots.

We also consider simplifications to this design problem that are obtained by specifying various dimensional parameters in the chain. In particular, we examine a three different nine position problems using polynomial continuation. Then we obtain analytical solutions for three design problems with six task positions, two with seven, and one with eight task positions.

Each set of design equations was analyzed to determine a linear product decomposition that provided a tighter bound the number of toots than the total degree. We found that these bounds were tight for the cases six through eight that we studied.

A Java implementation of the generalized eigenvalue solution for the eight task position problem has been integrated into our computer aided design tool SYNTHETICA to allow the designer to specify the spatial task, compute solutions, and then animate the resulting serial chains. A numerical example illustrating the results was provided.

# Acknowledgments

The authors gratefully acknowledge the support of the National Science Foundation.

# References

[1] Chen, P. and Roth, B., 1967, "Design Equations for Finitely and Infinitesimally Separated Position Synthesis of Binary Link and Combined Link Chains," *ASME J. Engineering for Industry* 91:209-219.

[2] Kim, H. S., Tsai L.-W., "Kinematics Synthesis of Spatial 3-RPS Parallel Manipulators," *ASME Design Engineering Technical Conference, Montreal, Canada, Sept.29-Oct.02, 2002.*

[3] Lee, E., and Mavroidis, D., 2002, "Solving the Geometric Design Problem of Spatial 3R Robot Manipulators Using Polynomial Homotopy Continuation," *ASME J. Mechanical Design*, 124(4):652-661.

[4] Manocha, D. , and Krishnan, S., 1996, "Solving Algebraic Systems using Matrix Computations," *ACM Sigsam Bulletin (Invited paper)*, 30(4), pp. 4-21, 1996.

[5] McCarthy, J. M., 2000, *Geometric Design of Linkages*, Springer-Verlag, New York, 2000.

[6] Morgan, A. P, Sommese, A. J, and Wampler, C. W., 1995, “A Product-Decomposition Bound for Bezout Numbers,” *SIAM J. of Numerical Analysis*, 32(4):1308-1325.

[7] Nielsen, J. and Roth, B., “Elimination Methods For Spatial Synthesis ” *Computational Kinematics* (eds. J.-P. Merlet and B. Ravani), Kluwer Academic Press, Netherlands, 1995.

[8] Su, H.-J., Wampler, C., McCarthy, J. M., “Geometric Design of Cylindric PRS Serial Chains,” submitted to *ASME Design Engineering Technical Conference, Chicago, Illinois, Sept. 02-06, 2003.*

[9] Su, H.-J., Collins, C., McCarthy, J. M., “An Extensible Java Applet for Spatial Linkage Synthesis,” *ASME Design Engineering Technical Conference, Montreal, Canada, Sept.29-Oct.02, 2002.*

[10] Verschelde, J, and Haegemans, A., 1993, “The GBQ-Algorithm for Constructing Start Systems of Homotopies for Polynomial Systems,” *SIAM J. Numerical Analysis*, 30(2):583-594.

[11] Verschelde, J., 1999, “Algorithm 795: PHCpack: A General-Purpose Solver for Polynomial Systems by Homotopy Continuation,” *ACM Trans Mathematical Software* 25(2):251-276.

[12] Wampler, C., Morgan, A. P., Sommese, A. J. 1990, “Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics,” *Journal of Mechanical Design* 112:59-68.