GEOMETRIC DESIGN OF RRP, RPR AND PRR SERIAL CHAINS

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Abstract

This paper formulates and solves the design equations for three degree-of-freedom spatial serial chains constructed with two revolute (R) joints and one prismatic (P) joint. Previous work has shown that equating the kinematics equations of the chain to a set of end-effector positions yields 24 equations in 25 unknowns, which means one of the design parameters can be chosen arbitrarily. Here it is shown that by specifying one of the directional parameters of the R-joints the design equations can be partitioned and solved separately. The derivation is presented in detail for the RRP chain, and the calculations are essentially identical for the the RPR and PRR chains. The special case in which the axes of the two R joints are perpendicular and intersect, forming a T-joint, is also presented. This chain has a spherical workspace is often used as the shoulder joint of a robot. Example calculations are presented.

1 Introduction

In this paper, we formulate the design equations for three-degree-of-freedom serial chains with the sequence of joints RRP, RPR, and PRR, where R represents a revolute, or hinged, joint, and P is a prismatic, or sliding, joint. The design equations are obtained by equating the kinematics equations of the chain to a set of specified positions for the end-effector. The goal is to determine the structural parameters of the chain that fit its workspace to a set of four arbitrarily specified positions of the end-effector.

We use a dual quaternion version of the kinematics equations of these chains to define the design equations. By selecting one of the direction parameters for one of the R-joints, we decouple these equations and obtain an analytical solution to these design equations. This strategy can be used to solve the design equations for all three cases.

An interesting special case of the RRP chain is the TP assembly, in which the two R-joints have axes that are perpendicular and intersect. The T-joint is often called a universal or U-joint. We show that this chain can be designed to reach three arbitrary end-effector positions using design equations have the same structure and are solved in the same way as the general case.

2 Literature Review

The design equations for serial chains consisting of two revolute joints and one prismatic joint were first formulated by Tsai [15] using the equivalent screw triangle. The case of the PRR robot was partially solved by Sandor et al. [13] for three task positions using loop equations. Their formulation had free parameters that were used to optimize the performance of the mechanism. Lee and Mavroidis (2002)[4] formulated and solved the design of the PRR spatial chain using its standard robotics kinematics equations [2]. They evaluate these equations on four specified displacements and obtain 24 equations in 25 unknown structural and joint parameters. By eliminating these parameters, they obtain a univariate polynomial of degree 30, which they report has 12 actual solutions and 18 extraneous roots.

Our dual quaternion synthesis approach was initially presented for the RPR robot in [9]. The dual quaternion form of the robot kinematics equations are identical to those obtained using successive screw displacements from a reference position (Gupta

1986[3], Tsai 1999[16]). It can also be shown to be equivalent to the "product of exponentials" formulation of the kinematic equations, see Murray et al. (1994)[8] and Park et al. (1995)[12]. The benefit of the dual quaternion formulation is the explicit specification of the Plücker coordinates of the joint axes. See Bottema and Roth (1979) [1] or McCarthy (1990) [7]. The application of this approach to the design of RPC and CRR robots can be found in Perez and McCarthy (2003ab)[10, 11].

The dual quaternion synthesis equations are closely related to the equivalent screw triangle formulation introduced by Tsai and Roth (1972) [14]. Tsai's dissertation [15] provides a large list of chains and associated design equations that can be obtained using his approach. We find that by using dual quaternions, we obtain the benefits of the equivalent screw triangle formulation combined with the ease of derivation of robot kinematics equations used by Mavroidis [6].

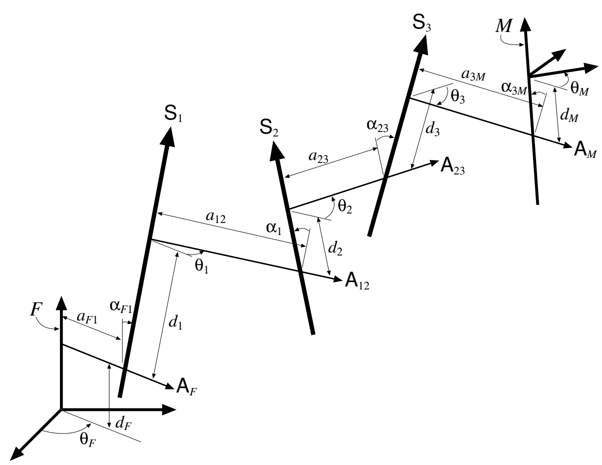

Figure 1: The parameters that define a serial chain with axes S_1, S_2 and S_3.

3 The Kinematics Equations

Consider the serial chain formed by three joint axes $\mathsf{S}_i, i = 1, 2, 3$ with the common normal lines A_{12} and A_{23} that link the first two, and second two of these axes, respectively. Figure 1 shows the skeleton of this chain positioned relative to a fixed frame F and with the frame M as its end-effector. The parameters $(\theta_i, d_i), i = 1, 2, 3$ define the movement at each joint axis and $(\alpha_{i,i+1}, a_{i,i+1}), i = 1, 2$ are the length and twist of the two links.

They are collectively known as the Denavit-Hartenberg parameters of the chain.

The kinematics equations for this serial chain equate the 4×4 homogeneous transformation $[D]$ between the end-effector frame M and the base frame F to the sequence of local coordinate transformations along the three joint axes (Craig 1989 [2]),

$$[D] = [G][Z(\theta_1, d_1)][X(\alpha_{12}, a_{12})][Z(\theta_2, d_2)][X(\alpha_{23}, a_{23})][Z(\theta_3, d_3)][H], \tag{1}$$

where $[Z(\cdot,\cdot)]$ is the 4×4 homogeneous transform that represents a screw displacement around and along the Z axis, and $[X(\cdot,\cdot)]$ is the screw displacement around and along the X axis.

The 4×4 matrix $[G]$ defines the position of the base of the chain relative to the fixed frame, and $[H]$ locates the end-effector relative to the last link frame.

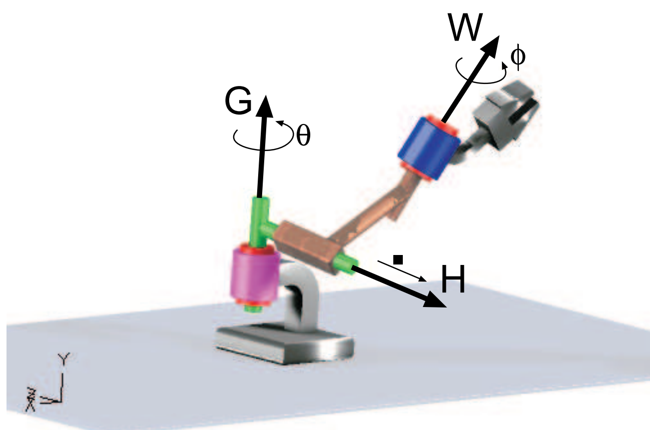

Figure 2: The RPR constrained serial robot.

The kinematics equations of the three serial chains RRP, RPR and RRP differ only in which of the joint parameters (θ_i, d_i) are considerd to be variable for each joint. The P joint only allows sliding which means that θ_i is constant and d_i is variable, while for the R joint it is θ_i that varies while d_i is constant. The three cases are shown in Table 1 .

3.1 Successive Screw Displacements

The usual kinematics equations of a serial chains are defined in terms of relative transformations along each of the links in the chain. These equations can be transformed

Case	Chain	Axis S_1	Axis S_2	Axis S_3
1	RRP	θ_1	θ_2	d_3
2	RPR	θ_1	d_2	θ_3
3	PRR	d_1	θ_2	θ_3

Table 1: The joint variables for the RRP, RPR and RRP serial chains.

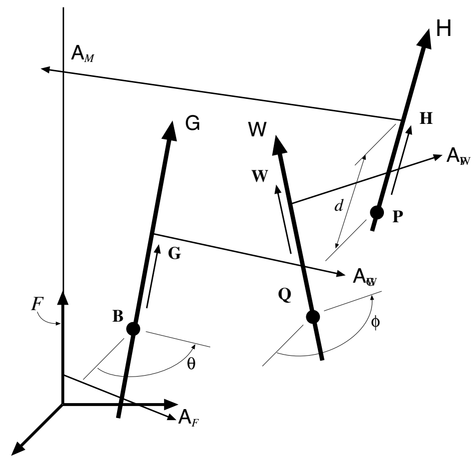

Figure 3: In the RRP chain translations along H, followed by rotations around W and then G, define the displacement of a frame coincident with fixed frame F to a new position.

into a sequence of screw displacements around axes with positions that are defined in the fixed frame. Tsai (1999)[16] calls this the method of successive screw displacements, and Gupta (1986)[3] calls it the "zero reference frame" formulation. It is also related to the "product of exponentials" formulation based on Lie Group methods. See Murray et al. (1994)[8], and Park et al. (1995)[12].

In the product of exponentials formulation of the kinematics equations of a serial chain, the instantaneous movement around each joint is defined as an element of the Lie Algebra $se(3)$, and assembled into a 4×4 matrix. Consider the line $L = \mathbf{C} + a\mathbf{S}$ in F, where a is a variable parameter, which is the screw axis of an instantaneous movement of a rigid body. If $\mathbf{w} = w\mathbf{S}$ is the angular velocity of this body around L and $\mathbf{v} = v\mathbf{S}$ is its linear velocity along L, then the Lie Algebra element representing this movement is given by the 4×4 matrix

$$[S] = \begin{bmatrix} w[\mathbf{S}] & w\mathbf{C} \times \mathbf{S} + v\mathbf{S} \\ 000 & 0 \end{bmatrix}. \tag{2}$$

The matrix $[\mathbf{S}]$ is the 3×3 cross product matrix constructed from the vector $\mathbf{S}$, that is $[\mathbf{S}]\mathbf{y} = \mathbf{S} \times \mathbf{y}$ for any vector $\mathbf{y}$. The 4×4 matrix transformation associated with $[S]$ is obtained as the matrix exponential,

$$[T] = e^{[S]t}, \tag{3}$$

which, for a screw axis fixed in F, has the angle $\psi = wt$ and slide $k = vt$ around and along L.

The two parameters ψ and k and the two vectors $\mathbf{S}$, and $\mathbf{C} \times \mathbf{S}$ completely define the displacement $[T]$. It is convenient to package these parameters using dual numbers so we have $\hat{\psi} = \psi + \epsilon k$ and $\mathsf{S} = \mathbf{S} + \epsilon \mathbf{C} \times \mathbf{S}$ in order to define the 4×4 *screw displacement* matrix $[T(\hat{\psi}, \mathsf{S})]$. See Bottema and Roth (1979)[1] or McCarthy (1990)[7] for a discussion of dual numbers and vectors constructed using the dual unit ϵ with the property that $\epsilon^2 = 0$. This dual vector algebra provide a convenient tool for manipulating the Plücker coordinates of lines.

The movement of the RRP serial chain can now be viewed as a consisting of a translation by d along $\mathsf{H} = \mathbf{H} + \epsilon \mathbf{P} \times \mathbf{H}$, followed by a by a rotation by ϕ about $\mathsf{W} = \mathbf{W} + \epsilon \mathbf{Q} \times \mathbf{W}$, and finally a rotation by θ around $\mathsf{G} = \mathbf{G} + \epsilon \mathbf{B} \times \mathbf{G}$. See Figure 3. The kinematics equations of this chain can be written as the product of successive screw displacements,

$$[T(\hat{\psi}, \mathsf{S})] = [T(\theta, \mathsf{G})][T(\phi, \mathsf{W})][T(d, \mathsf{H})]. \tag{4}$$

Because each of these screw displacements is the exponential of the Lie algebra element, this equation is equivalent to the "product of exponentials" formulation (Murray, et al. 1994[8]).

4 Dual Quaternion Kinematics Equations

For the geometric design problem, we are given screw axes and associated rotation angles and slides for a set of n end-effector positions, and we use the kinematics equations to compute the axes G, W and H. It is convenient for this purpose to have the kinematics equations formulated directly in screw parameters of the serial chain rather than as 4×4 homogeneous transforms. This is achieved by using elements of the Clifford algebra of spatial displacements, known as *dual quaternions* (Yang 1964 [17], Bottema and Roth 1979 [1], and McCarthy 1990 [7]).

A dual quaternion is the combination of a dual number $\hat{a} = a + \epsilon a^\circ$ and a dual vector $\mathsf{A} = \mathbf{A} + \epsilon\mathbf{A}^\circ$ that forms the hypercomplex number $\hat{A} = \hat{a} + \mathsf{A}$. The product of two dual quaternions is defined to be

$$\hat{A}\hat{B} = (\hat{a} + \mathsf{A})(\hat{b} + \mathsf{B}) = (\hat{a}\hat{b} - \mathsf{A}\cdot\mathsf{B}) + (\hat{a}\mathsf{B} + \hat{b}\mathsf{A} + \mathsf{A}\times\mathsf{B}), \tag{5}$$

where the dot and cross represent the usual scalar and vector products, but now applied to dual vectors.

A general screw displacement $[T(\hat{\psi}, \mathsf{S})]$ is identified with the dual quaternion

$$\hat{S}(\hat{\psi}) = \cos\frac{\hat{\psi}}{2} + \sin\frac{\hat{\psi}}{2}\mathsf{S}, \tag{6}$$

where the sine and cosine of a dual angle are defined by

$$\cos\frac{\hat{\psi}}{2} = \cos\frac{\psi}{2} - \epsilon\frac{d}{2}\sin\frac{\psi}{2}, \quad \text{and} \quad \sin\frac{\hat{\psi}}{2} = \sin\frac{\psi}{2} + \epsilon\frac{d}{2}\cos\frac{\psi}{2}. \tag{7}$$

The special cases of dual quaternions associated with revolute and prismatic joints are defined to be, respectively, a pure rotation of θ around an axis $\mathsf{S} = \mathbf{S} + \epsilon\mathbf{S}^\circ$ and a pure translation by d along S, that is

$$\begin{aligned} \text{Revolute Joint:} \quad & \hat{S}(\theta) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathsf{S}, \\ \text{Prismatic Joint:} \quad & \hat{S}(d) = 1 + \epsilon\frac{d}{2}\mathbf{S}. \end{aligned} \tag{8}$$

Recall that for the Plücker coordinates joint axis $\mathbf{S}^\circ = \mathbf{B}\times\mathbf{S}$, where $\mathbf{B}$ is a point on the axis, we can compute $\mathbf{B}$ using the formula

$$\mathbf{B} = \frac{\mathbf{J}\times\mathbf{J}^\circ}{\mathbf{J}\cdot\mathbf{J}}. \tag{9}$$

Notice that the coordinates for this point drop out of the dual quaternion for a prismatic joint, which shows that this displacement does not depend on the location of the axis, only its direction.

It is now easy to construct the dual quaternion kinematics equations for the RRP chain as

$$\text{RRP:}\quad \hat{S}(\hat{\psi}) = \hat{G}(\theta)\hat{W}(\phi)\hat{H}(d) = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathsf{G})(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\mathsf{W})(1 + \epsilon\frac{d}{2}\mathbf{H}). \quad (10)$$

Similarly, the kinematics equations for the RPR and RRP chains are given by,

$$\text{RPR:}\quad \hat{S}(\hat{\psi}) = \hat{G}(\theta)\hat{H}(d)\hat{W}(\phi) = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathsf{G})(1 + \epsilon\frac{d}{2}\mathbf{H})(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\mathsf{W}), \quad (11)$$

and

$$\text{PRR:}\quad \hat{S}(\hat{\psi}) = \hat{H}(d)\hat{G}(\theta)\hat{W}(\phi) = (1 + \epsilon\frac{d}{2}\mathbf{H})(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathsf{G})(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\mathsf{W}). \quad (12)$$

Four independent parameters define each axis of the two revolute joints G and W, and two parameters define the direction of the prismatic joint H. The 10 parameters defining the joint axes plus six parameters to define the zero position of the end-effector yield 16 parameters that define the initial configuration for each chain.

5 The Design Equations

In order to formulate the design equations for the RRP chain, we equate the dual quaternion kinematics equations to three relative positions, which yields 18 independent design equations in the following unknowns: 10 parameters to define the joint axes, plus three joint variables to reach each of the positions. This is a total of 18 equations in 19 unknowns, thus one of the unknown parameters may be chosen arbitrarly. We choose to fix one of the components of the directions $\mathbf{G}$ or $\mathbf{W}$ and obtain an important simplification to the design equations, that applies to each of the RRP, RPR and PRR chains. This can be compared to the 4×4 homogeneous formulation of Lee and Mavroidis (2002)[4], which yields 24 design equations in 25 unknowns, in which they choose to specify a parameter of the prismatic joint.

Consider the set of four specified goal positions $[T(\hat{\psi}_i, \mathsf{S}_i)], i = 1, 2, 3, 4$ which define the dual quaternions $\hat{P}_i = \cos\frac{\hat{\psi}_i}{2} + \sin\frac{\hat{\psi}_i}{2}\mathsf{S}_i$. The design equations are created by equating the kinematics equations to each one of the task positions. We choose the zero position of the end-effector to be the first goal position. This allows us to formulate both the kinematics equations and the task positions relative to the first goal position by computing $\hat{P}_i\hat{P}_1^{-1} = \hat{P}_{1i}(\hat{\psi}_{1i})$, which yields,

$$
\begin{aligned}
\text{RRP:}\quad \hat{P}_{12}(\hat{\psi}_{12}) =& \hat{G}(\theta_{12})\hat{W}(\phi_{12})\hat{H}(d_{12}),\\
\hat{P}_{13}(\hat{\psi}_{13}) =& \hat{G}(\theta_{13})\hat{W}(\phi_{13})\hat{H}(d_{13}),\\
\hat{P}_{14}(\hat{\psi}_{14}) =& \hat{G}(\theta_{14})\hat{W}(\phi_{14})\hat{H}(d_{14}).
\end{aligned} \tag{13}
$$

We use these equations to design the PRR chain. Design equations for the RPR and RRP chains are obtained in the same way to obtain

$$
\begin{aligned}
\text{RPR}: \quad & \hat{P}_{1i}(\hat{\psi}_{1i}) = \hat{G}(\theta_{1i})\hat{H}(d_{1i})\hat{W}(\phi_{1i}), \quad i = 2, 3, 4,\\
\text{PRR}: \quad & \hat{P}_{1i}(\hat{\psi}_{1i}) = \hat{H}(d_{1i})\hat{G}(\theta_{1i})\hat{W}(\phi_{1i}), \quad i = 2, 3, 4.
\end{aligned} \tag{14}
$$

In order to use these design equations, we expand the dual quaternion $\hat{P} = \hat{p} + \mathsf{P} = (p + \epsilon p^\circ) + (\mathbf{P} + \epsilon\mathbf{P}^\circ)$ and collect the real and dual components to obtain $\hat{P} = (p+\mathbf{P}) + \epsilon(p^\circ + \mathbf{P}^\circ) = P + \epsilon P^\circ$. This allows us to write the typical design equation for the three chains in the form,

$$
\begin{aligned}
\text{RRP:}\quad P_{1i} + \epsilon P_{1i}^\circ =& GWH + \epsilon(GWH^\circ + G^\circ WH + GW^\circ H),\\
\text{PRR:}\quad P_{1i} + \epsilon P_{1i}^\circ =& HGW + \epsilon(H^\circ GW + HG^\circ W + HGW^\circ),\\
\text{RPR:}\quad P_{1i} + \epsilon P_{1i}^\circ =& GHW + \epsilon(GH^\circ W + G^\circ HW + GHW^\circ).
\end{aligned} \tag{15}
$$

The quaternion factors in these equations are given by

$$
\begin{aligned}
G =& \cos\frac{\theta_{1i}}{2} + \sin\frac{\theta_{1i}}{2}\mathbf{G}, \quad G^\circ = \sin\frac{\theta_{1i}}{2}\mathbf{G}^\circ,\\
W =& \cos\frac{\phi_{1i}}{2} + \sin\frac{\phi_{1i}}{2}\mathbf{W}, \quad W^\circ = \sin\frac{\phi_{1i}}{2}\mathbf{W}^\circ,\\
H =& 1, \quad H^\circ = \frac{d_{1i}}{2}\mathbf{H}.
\end{aligned} \tag{16}
$$

Notice that because $H = 1$, Eq.(15) becomes

$$
\begin{aligned}
\text{RRP}: \quad & P_{1i} + \epsilon P_{1i}^\circ = GW + \epsilon(GWH^\circ + G^\circ W + GW^\circ),\\
\text{RPR}: \quad & P_{1i} + \epsilon P_{1i}^\circ = GW + \epsilon(GH^\circ W + G^\circ W + GW^\circ),\\
\text{PRR}: \quad & P_{1i} + \epsilon P_{1i}^\circ = GW + \epsilon(H^\circ GW + G^\circ W + GW^\circ).
\end{aligned} \tag{17}
$$

It is important to notice that the real parts of these three equations are identical. Furthermore, the dual parts differ only in the order of the quaternion multiplication of one term. Thus, the design equations for the three chains differ only by the signs of the dual part.

Now let $P_{1i} = p_{1i} + \mathbf{P}_{1i}$ and $\mathsf{P}^\circ_{1i} = p^\circ_{1i} + \mathbf{P}^\circ_{1i}$, and expand the quaternion products in Eq.(17) for the RRP robot, see Figure 4. The real part is the same for all three chains and is given by

$$\begin{aligned} p_{1i} =& \cos\frac{\theta_{1i}}{2}\cos\frac{\phi_{1i}}{2} - \sin\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\mathbf{G}\cdot\mathbf{W}, \\ \mathbf{P}_{1i} =& \sin\frac{\theta_{1i}}{2}\cos\frac{\phi_{1i}}{2}\mathbf{G} + \sin\frac{\phi_{1i}}{2}\cos\frac{\theta_{1i}}{2}\mathbf{W} + \sin\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\mathbf{G}\times\mathbf{W}, \\ & \qquad i = 2,3,4. \end{aligned} \tag{18}$$

Expand the dual part for the RRP chain,

$$\begin{aligned} p^\circ_{1i} =& -\frac{d_{1i}}{2}\sin\frac{\theta_{1i}}{2}\cos\frac{\phi_{1i}}{2}\mathbf{G}\cdot\mathbf{H} - \frac{d_{1i}}{2}\cos\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\mathbf{W}\cdot\mathbf{H} \\ & - \sin\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}(\mathbf{G}\cdot\mathbf{W}^\circ + \mathbf{G}^\circ\cdot\mathbf{W}) - \frac{d_{1i}}{2}\sin\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\mathbf{G}\times\mathbf{W}\cdot\mathbf{H}, \\ \mathbf{P}^\circ_{1i} =& \sin\frac{\theta_{1i}}{2}\cos\frac{\phi_{1i}}{2}\mathbf{G}^\circ + \cos\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\mathbf{W}^\circ + \sin\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}(\mathbf{G}\times\mathbf{W}^\circ + \mathbf{G}^\circ\times\mathbf{W}) \\ & + \frac{d_{1i}}{2}\cos\frac{\theta_{1i}}{2}\cos\frac{\theta_{1i}}{2}\mathbf{H} + \frac{d_{1i}}{2}\sin\frac{\theta_{1i}}{2}\cos\frac{\phi_{1i}}{2}\mathbf{G}\times\mathbf{H} + \frac{d_{1i}}{2}\cos\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\mathbf{W}\times\mathbf{H} \\ & + \frac{d_{1i}}{2}\sin\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}((\mathbf{G}\times\mathbf{W})\times\mathbf{H} - (\mathbf{G}\cdot\mathbf{W})\mathbf{H}), \quad i = 2,3,4. \end{aligned} \tag{19}$$

The RPR and PRR chains have essentially the same expanded dual part, except the order of some products with the term $\mathbf{H}$ change, which creates some sign changes.

Notice now that the parameters p_{1i} and p°_{1i} and the vectors $\mathbf{P}_{1i}$ and $\mathbf{P}^\circ_{1i}$ are known, if we impose them to be the four specified task positions. The equations (18) and (19) can be solved for the unknown vectors $\mathbf{H}$, $\mathbf{G}$, $\mathbf{W}$, $\mathbf{G}^\circ$ and $\mathbf{W}^\circ$ and the joint parameters d_{1i}, θ_{1i} and ϕ_{1i}, $i = 2,3,4$.

This design problem consists of 24 equations, only 18 of which are independent, with 19 unknowns, of which 10 are structural parameters and 9 are relative joint variables. In what follows, we obtain an analytical solution by specifying one structural parameter.

6 Simplifying the Design Equations

In this section we perform what we call *inverse kinematics elimination*, which aims to simplify the design equations by eliminating the joint variables. The polynomial system

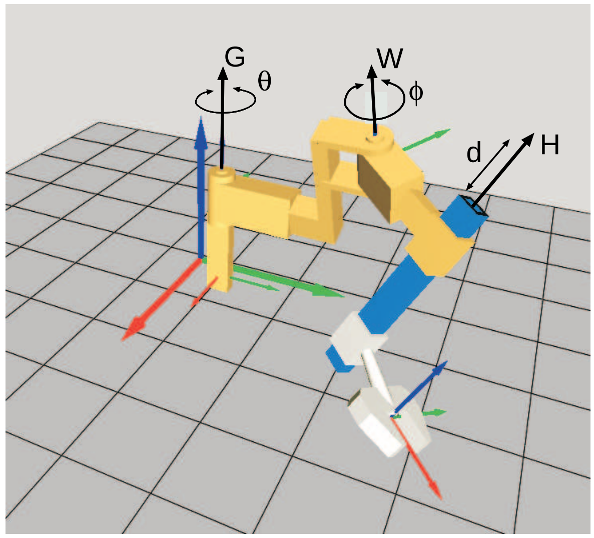

Figure 4: The RRP robot

obtained after this elimination can be further simplified using resultant methods, to obtain a univariate polynomial. We do this simplification for the RRP chain; the procedure is identical for the RPR and PRR chains.

The coefficients of each of the design equations (18) and (19) are constructed from sine and cosines of the joint variables in a pattern that is captured in the following dual quaternion

$$\hat{R} = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\vec{\imath})((\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\vec{\jmath})(1 + \epsilon\frac{d}{2}\vec{k}), \tag{20}$$

where $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ are unit vectors along the X, Y and Z axes of F. Expanding this equation, we obtain $\hat{R} = (r + \mathbf{R}) + \epsilon(r^\circ + \mathbf{R}^\circ)$, which we write in vector form as

$$\hat{R} = R + \epsilon R^\circ = \begin{Bmatrix} \mathbf{R} \\ r \end{Bmatrix} + \epsilon \begin{Bmatrix} \mathbf{R}^\circ \\ r^\circ \end{Bmatrix} = \begin{Bmatrix} \sin\frac{\theta}{2}\cos\frac{\phi}{2} \\ \cos\frac{\theta}{2}\sin\frac{\phi}{2} \\ \sin\frac{\theta}{2}\sin\frac{\phi}{2} \\ \cos\frac{\theta}{2}\cos\frac{\phi}{2} \end{Bmatrix} + \epsilon \begin{Bmatrix} \frac{d}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} \\ -\frac{d}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2} \\ \frac{d}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} \\ -\frac{d}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2} \end{Bmatrix}. \tag{21}$$

If we call $\hat{R}(\theta_{1i}, \phi_{1i}, d_{1i}) = \hat{R}_{1i}$, we can now assemble the design equations, (18) and (19), into the 8×8 matrix equation

$$\begin{Bmatrix} P_{1i} \\ P^\circ_{1i} \end{Bmatrix} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{Bmatrix} R_{1i} \\ R^\circ_{1i} \end{Bmatrix}, \tag{22}$$

where A, B and C are 4×4 matrices given by

$$[A] = \begin{bmatrix} \mathbf{G} & \mathbf{W} & \mathbf{G}\times\mathbf{W} & \vec{0} \\ 0 & 0 & -\mathbf{G}\cdot\mathbf{W} & 1 \end{bmatrix}, \quad [B] = \begin{bmatrix} \mathbf{G}^\circ & \mathbf{W}^\circ & \mathbf{G}^\circ\times\mathbf{W}+\mathbf{G}\times\mathbf{W}^\circ & \vec{0} \\ 0 & 0 & -(\mathbf{G}^\circ\cdot\mathbf{W}+\mathbf{G}\cdot\mathbf{W}^\circ) & 0 \end{bmatrix},$$

and

$$[C] = \begin{bmatrix} \mathbf{W}\times\mathbf{H} & -\mathbf{G}\times\mathbf{H} & \mathbf{H} & -(\mathbf{G}\times\mathbf{W})\times\mathbf{H}+(\mathbf{G}\cdot\mathbf{W})\mathbf{H} \\ -\mathbf{W}\cdot\mathbf{H} & \mathbf{G}\cdot\mathbf{H} & 0 & (\mathbf{G}\times\mathbf{W})\cdot\mathbf{H} \end{bmatrix}. \tag{23}$$

In order to solve for the joint variables, the matrix equations can be inverted symbolically to obtain

$$\begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix} \begin{Bmatrix} P_{1i} \\ P^\circ_{1i} \end{Bmatrix} = \begin{Bmatrix} R_{1i} \\ R^\circ_{1i} \end{Bmatrix}, \tag{24}$$

where the inverse of the matrices $[A]$ and $[C]$ are given in terms of the unknown vectors as

$$[A]^{-1} = \frac{1}{(\mathbf{G}\times\mathbf{W})^2} \begin{bmatrix} \mathbf{W}\times(\mathbf{G}\times\mathbf{W})^T & 0 \\ -\mathbf{G}\times(\mathbf{G}\times\mathbf{W})^T & 0 \\ \mathbf{G}\times\mathbf{W}^T & 0 \\ (\mathbf{G}\cdot\mathbf{W})\mathbf{G}\times\mathbf{W}^T & (\mathbf{G}\times\mathbf{W})^2 \end{bmatrix} \tag{25}$$

and,

$$[C]^{-1} = \frac{1}{(\mathbf{G}\times\mathbf{W})^2} \begin{bmatrix} -(\mathbf{G}\times(\mathbf{G}\times\mathbf{W}))\times\mathbf{H}^T & (\mathbf{G}\times(\mathbf{G}\times\mathbf{W}))\cdot\mathbf{H} \\ (\mathbf{W}\times(\mathbf{W}\times\mathbf{G}))\times\mathbf{H}^T & -(\mathbf{W}\times(\mathbf{W}\times\mathbf{G}))\cdot\mathbf{H} \\ (\mathbf{G}\cdot\mathbf{W})(\mathbf{G}\times\mathbf{W})\times\mathbf{H}^T+(\mathbf{G}\times\mathbf{W})^2\mathbf{H}^T & -(\mathbf{G}\cdot\mathbf{W})\mathbf{H}\cdot(\mathbf{G}\times\mathbf{W}) \\ -(\mathbf{G}\times\mathbf{W})\times\mathbf{H}^T & (\mathbf{G}\times\mathbf{W})\cdot\mathbf{H} \end{bmatrix}. \tag{26}$$

Equation (24) is actually the inverse kinematics solution for the RRP chain. If the dimensions of the chain are specified, then $[A]$, $[B]$ and $[C]$ are known, and this equation determines the joint parameters d_{1i}, θ_{1i} and ϕ_{1i} for an end-effector location $\hat{P}_{1i}$.

We now use the inverse kinematics solution (24) to eliminate the joint variables from our design equations. The first four components represent the rotation component of the movement of the chain,

$$(\mathbf{G}\times\mathbf{W})^2\begin{Bmatrix}\sin\frac{\theta_{1i}}{2}\cos\frac{\phi_{1i}}{2}\\ \cos\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\\ \sin\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\\ \cos\frac{\theta_{1i}}{2}\cos\frac{\phi_{1i}}{2}\end{Bmatrix}=\begin{Bmatrix}\mathbf{W}\times(\mathbf{G}\times\mathbf{W})\cdot\mathbf{P}_{1i}\\ -\mathbf{G}\times(\mathbf{G}\times\mathbf{W})\cdot\mathbf{P}_{1i}\\ \mathbf{G}\times\mathbf{W}\cdot\mathbf{P}_{1i}\\ (\mathbf{G}\cdot\mathbf{W})(\mathbf{G}\times\mathbf{W})\cdot\mathbf{P}_{1i}+p_{1i}(\mathbf{G}\times\mathbf{W})^2\end{Bmatrix}. \tag{27}$$

The second four components have longer expressions so we write them in the form

$$\begin{Bmatrix}\frac{d_{1i}}{2}\cos\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\\ -\frac{d_{1i}}{2}\sin\frac{\theta_{1i}}{2}\cos\frac{\phi_{1i}}{2}\\ \frac{d_{1i}}{2}\cos\frac{\theta_{1i}}{2}\cos\frac{\phi_{1i}}{2}\\ -\frac{d_{1i}}{2}\sin\frac{\theta_{1i}}{2}\sin\frac{\phi_{1i}}{2}\end{Bmatrix}=-([C]^{-1}[B][A]^{-1})\begin{Bmatrix}\mathbf{P}_{1i}\\ p_{1i}\end{Bmatrix}+[C]^{-1}\begin{Bmatrix}\mathbf{P}^\circ_{1i}\\ p^\circ_{1i}\end{Bmatrix}. \tag{28}$$

The structure of the dual quaternion $\hat{R}$, Eq.(21), allows us to identify three relations among its components,

$$\begin{aligned}\mathcal{R}:&\quad R_xR_y-R_zr=0,\\ \mathcal{S}:&\quad R_xR^\circ_x+R_yR^\circ_y=0,\\ \mathcal{T}:&\quad R_zR^\circ_z+rr^\circ=0,\end{aligned} \tag{29}$$

where $\mathbf{R}=(R_x,R_y,R_z)$ and $\mathbf{R}^\circ=(R^\circ_x,R^\circ_y,R^\circ_z)$. We use these relations, which are free of joint variables, to obtain the polynomial design equations that we then solve by elimination.

7 Solving the Design Equations

In this section, we substitute the values of the joint variables from the inverse kinematics equations (24) in the relations (29) to formulate polynomial design equations, which we then solve by elimination. We start with the relation $\mathcal{R}=0$ which depends only on the directions, $\mathbf{G}$ and $\mathbf{W}$, of the two revolute axes. Substitute (27) into this equation, and manipulate the results to obtain

$$\mathcal{R}:\quad (p_{1i}\mathbf{G}\cdot\mathbf{P}_{1i}\times\mathbf{W}+\mathbf{G}\cdot(\mathbf{P}_{1i}\times(\mathbf{P}_{1i}\times\mathbf{W})))(\mathbf{G}\times\mathbf{W})^2=0. \tag{30}$$

If we require that $\mathbf{G}\times\mathbf{W}\neq 0$, which restricts the pair of revolute joints from generating a pure planar movement, then for the three relative task positions, we obtain

$$\mathcal{R}_{1i}:\quad p_{1i}\mathbf{G}\cdot\mathbf{P}_{1i}\times\mathbf{W}+\mathbf{G}\cdot(\mathbf{P}_{1i}\times(\mathbf{P}_{1i}\times\mathbf{W}))=0,\quad i=2,3,4. \tag{31}$$

It is useful to notice that this equation is bilinear in the components of $\mathbf{G}$ and $\mathbf{W}$.

Recall that the p_{1i} and $\mathbf{P}_{1i}$ are known, so this is three equations in the six unknown $\mathbf{G} = (g_x, g_y, g_z)$ and $\mathbf{W} = (w_x, w_y, w_z)$. The magnitude of these vectors does not affect the ultimate properties of the chain, therefore, we can choose arbitrary vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ and scalars e_1 and e_2, and introduce the normalizing conditions

$$\begin{aligned} \mathcal{C}_1 : &\quad \mathbf{m}_1 \cdot \mathbf{G} = e_1, \\ \mathcal{C}_2 : &\quad \mathbf{m}_2 \cdot \mathbf{W} = e_2. \end{aligned} \tag{32}$$

The result is five equations in six unknowns. To solve these equations, we have the freedom to specify an arbitrary relationship among these six coordinates. For our purposes, set $w_y = 0$.

Substitute $w_y = 0$ into (31) and use $\mathcal{C}_1$ to eliminate g_z and $\mathcal{C}_2$ to eliminate w_z, then assemble the polynomials $\mathcal{R}_{1i}$ into the matrix equation,

$$\begin{bmatrix} a_{11}g_x + b_{11}g_y + c_{11} & a_{12}g_x + b_{12}g_y + c_{12} \\ a_{21}g_x + b_{21}g_y + c_{21} & a_{22}g_x + b_{22}g_y + c_{22} \\ a_{31}g_x + b_{31}g_y + c_{31} & a_{32}g_x + b_{32}g_y + c_{32} \end{bmatrix} \begin{Bmatrix} w_x \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{33}$$

The terms a_{ij}, b_{ij} and c_{ij} are constants determined by the task positions and the normalizing conditions. This matrix equation has a solution only if the rank of the coefficient matrix is 1. To achieve this we set the three 2×2 minors to zero. The result is the three quadratic polynomials in the components g_x, g_y of $\mathbf{G}$,

$$\mathcal{D}_j : \quad d_{j0}g_y^2 + (d_{j1}g_x + d_{j2})g_y + (d_{j3}g_x^2 + d_{j4}g_x + d_{j5}) = 0, \quad i = 1, 2, 3. \tag{34}$$

Assemble these minors into the matrix equation

$$\begin{bmatrix} d_{10} & d_{11}g_x + d_{12} & d_{13}g_x^2 + d_{14}g_x + d_{15} \\ d_{20} & d_{21}g_x + d_{22} & d_{23}g_x^2 + d_{24}g_x + d_{25} \\ d_{30} & d_{31}g_x + d_{32} & d_{33}g_x^2 + d_{34}g_x + d_{35} \end{bmatrix} \begin{Bmatrix} g_y^2 \\ g_y \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{35}$$

which has a solution if its determinant is zero. This yields a cubic polynomial in g_x. The three roots for g_x of this polynomial allow us to solve for g_y in (35), which in turn allows us to solve for w_x in (33). Thus, we obtain as many as three solutions for the directions $\mathbf{G}$ and $\mathbf{W}$ in the RRP chain. Notice that this result applies to the RPR and PRR chains as well, because the equations (31) are the same for all three chains.

The ability to solve for the directions $\mathbf{G}$ and $\mathbf{W}$ independently simplifies the solution of the remaining design equations. In particular, in Eq.(28) the matrix $[A]^{-1}$ is now constant, $[C]^{-1}$ is linear in the components $\mathbf{H} = (h_x, h_y, h_z)$ and $[B]$ is linear in the components $\mathbf{G}^\circ = (g_x^\circ, g_y^\circ, g_z^\circ)$ and $\mathbf{W}^\circ = (w_x^\circ, w_y^\circ, w_z^\circ)$. Substitute this into the two relations $\mathcal{S}$ and $\mathcal{T}$ for each of the three relative positions to obtain six equations of the form,

$$\begin{aligned}
\mathcal{S}_{1i}: \quad & h_x(\mathbf{A}_{xj}\cdot\mathbf{G}^\circ + \mathbf{B}_{xj}\cdot\mathbf{W}^\circ + c_{xj}) + h_y(\mathbf{A}_{yj}\cdot\mathbf{G}^\circ + \mathbf{B}_{yj}\cdot\mathbf{W}^\circ + c_{yj}) \\
& \qquad + h_z(\mathbf{A}_{zj}\cdot\mathbf{G}^\circ + \mathbf{B}_{zj}\cdot\mathbf{W}^\circ + c_{zj}) = 0, \quad i = 2,3,4, j = i-1, \\
\mathcal{T}_{1i}: \quad & h_x(\mathbf{A}_{xj}\cdot\mathbf{G}^\circ + \mathbf{B}_{xj}\cdot\mathbf{W}^\circ + c_{xj}) + h_y(\mathbf{A}_{yj}\cdot\mathbf{G}^\circ + \mathbf{B}_{yj}\cdot\mathbf{W}^\circ + c_{yj}) \\
& \qquad + h_z(\mathbf{A}_{zj}\cdot\mathbf{G}^\circ + \mathbf{B}_{zj}\cdot\mathbf{W}^\circ + c_{zj}) = 0, \quad i = 2,3,4, j = i+2. \qquad (36)
\end{aligned}$$

The coefficient vectors $\mathbf{A}_{xj}$, $\mathbf{A}_{yj}$, $\mathbf{A}_{zj}$ and $\mathbf{B}_{xj}$, $\mathbf{B}_{yj}$, $\mathbf{B}_{zj}$, as well as the terms c_{xj}, c_{yj}, c_{zj} are all constants determined by the specified task positions and the values of $\mathbf{G}$ and $\mathbf{W}$. Notice that these equations are bilinear in $\mathbf{H}$, and the combination of $\mathbf{G}^\circ$ and $\mathbf{W}^\circ$.

We have additional constraints on the unknown vectors $\mathbf{G}^\circ$, $\mathbf{W}^\circ$ and $\mathbf{H}$. First, the direction of $\mathbf{H}$ is important, not its magnitude therefore we can choose arbitrary parameters $\mathbf{m}_3$ and e_3, so $\mathbf{m}_3\cdot\mathbf{h} = e_3$. Next, the vectors $\mathbf{G}^\circ$ and $\mathbf{W}^\circ$ must satisfy the Plucker conditions in order to define a line, and we have the three linear constraints

$$\begin{aligned}
\mathcal{C}_3: \quad & \mathbf{m}_3\cdot\mathbf{h} = e_3, \\
\mathcal{C}_4: \quad & \mathbf{G}\cdot\mathbf{G}^\circ = 0, \quad \text{and} \\
\mathcal{C}_5: \quad & \mathbf{W}\cdot\mathbf{W}^\circ = 0. \qquad (37)
\end{aligned}$$

We use these equations to eliminate the parameters h_z, g_z°, and g_z° in Eq.(36), in order to obtain the 6×5 matrix equation

$$\begin{bmatrix}
a_{11}h_x + b_{11}h_y + c_{11} & a_{12}h_x + b_{12}h_y + c_{12} & \dots & a_{15}h_x + b_{15}h_y + c_{15} \\
a_{21}h_x + b_{21}h_y + c_{21} & a_{22}h_x + b_{22}h_y + c_{22} & \dots & a_{25}h_x + b_{25}h_y + c_{25} \\
\vdots & \vdots & \dots & \vdots \\
a_{61}h_x + b_{61}h_y + c_{61} & a_{62}h_x + b_{62}h_y + c_{62} & \dots & a_{65}h_x + b_{65}h_y + c_{45}
\end{bmatrix}
\begin{Bmatrix} g_x^\circ \\ g_y^\circ \\ w_x^\circ \\ w_y^\circ \\ 1 \end{Bmatrix}
= \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}. \qquad (38)$$

This equation can be solved for the four parameters g_x°, g_y°, w_x°, and w_y° , if we force the six 5×5 minors to be zero, so the system has rank four.

The six minors in (38) yield six polynomials of degree five in h_x and h_y. Relabel these variables so $h_x = x$ and $h_y = y$, collect coefficients of x, and construct the 6×6 matrix equation

$$\begin{bmatrix} d_{10} & d_{11}(y) & d_{12}(y) & d_{13}(y) & d_{14}(y) & d_{15}(y) \\ d_{20} & d_{21}(y) & d_{22}(y) & d_{23}(y) & d_{24}(y) & d_{25}(y) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{60} & d_{61}(y) & d_{62}(y) & d_{63}(y) & d_{64}(y) & d_{65}(y) \end{bmatrix} \begin{Bmatrix} x^5 \\ x^4 \\ \vdots \\ x \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix}. \tag{39}$$

The matrix $[D(y)] = [d_{ij}(y)]$ formed by these coefficients has elements that are polynomials of degree j in y. We can set $\det[D] = 0$ to obtain a univariate polynomial of degree 15. A solution $y = h_y$ of this polynomial allows us to compute an associated $x = h_x$ in (39). These values are substituted into (38) to determine corresponding values for g_x°, g_y°, w_x°, and w_y°. Thus, completing the solution of the design equations for the RRP chain.

Manocha (1998) [5] provides a convenient way to find the values y such that $\det[D(y)] = 0$ by computing the eigenvalues of a matrix constructed from $[D(y)]$. The first step is to expand $[D(y)]$ as a matrix polynomial in y, that is

$$[D(y)] = [M_0] + [M_1]y + [M_2]y^2 + [M_3]y^3 + [M_4]y^4 + [M_5]y^5. \tag{40}$$

Notice that the structure of the coefficients $d_{ij}(y)$ forces the first i columns of the 6×6 matrices $[M_i]$ to be zero. This means only $[M_0]$ is non-singular.

We now transform $[D(y)]$ into a monic polynomial. However, because $[M_0]$ is the only invertable coefficient matrix, we must transform coordinates so $z = 1/y$, and define

$$[D(z)] = [M_0]z^5 + [M_1]z^4 + [M_2]z^3 + [M_3]z^2 + [M_4]z + [M_5]. \tag{41}$$

Multiply by $[M_0]^{-1}$ to obtain

$$[\bar{D}(z)] = [I]z^5 + [\bar{M}_1]z^4 + [\bar{M}_2]z^3 + [\bar{M}_3]z^2 + [\bar{M}_4]z + [\bar{M}_5], \tag{42}$$

where $[I]$ is the 6×6 identity matrix and $[\bar{M}_i] = [M_0]^{-1}[M_i]$.

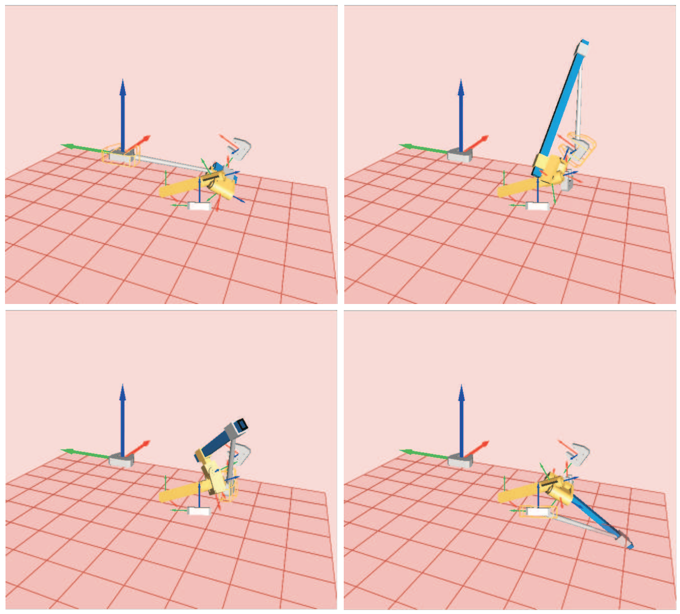

Figure 5: One of the RRP solutions reaching positions 1, 2, 3 and 4

The roots of $\det[\bar{D}(z)] = 0$ are obtained as the eigenvalues of the 30×30 matrix

$$[C] = \begin{bmatrix} 0 & [I] & 0 & 0 & 0 \\ 0 & 0 & [I] & 0 & 0 \\ 0 & 0 & 0 & [I] & 0 \\ 0 & 0 & 0 & 0 & [I] \\ -[\bar{M}_5] & -[\bar{M}_4] & -[\bar{M}_3] & -[\bar{M}_2] & -[\bar{M}_1] \end{bmatrix}. \quad (43)$$

This result and a general fomulation that does not rely on a monic structure fpr the matrix polynomial is discussed in detail in [5].

We solved this problem both using the resultant polynomial obtained from $\det[D(y)]$ and by computing the 15 non-zero eigenvalues of $[C]$ using Mathematica. The computations in both cases took less than 1 second and were identical. In completing the solution, we found that only four of the 15 roots yielded physical RRP chains. This combines with the three roots of the directions equations to yield 12 solutions.

8 Example RRP Synthesis

For this example, we defined a set of four random positions, see Table 8. We obtained twelve real roots and present one of the solutions in Table 8 and Figure 5.

Position	*Axis*	*Rot.*	*Trans.*
position 1	$(1.0, 0.0, 0.0) + \epsilon(0.0, 0.0, 0.0)$	0^o	0
position 2	$(0.51, 0.73, -0.45) + \epsilon(-0.60, 1.58, 1.86)$	262.0^o	-0.08
position 3	$(-0.68, -0.55, -0.49) + \epsilon(0.91, 0.27, -1.54)$	123.1^o	0.48
position 4	$(-0.26, 0.87, 0.42) + \epsilon(0.09, 0.32, -0.62)$	27.9^o	-2.32

Table 2: The goal positions

Joint Axis	*Direction*	*Moment*
G	$(0, -0.90, 0.44)$	$(-1.60, -0.60, -1.20)$
W	$(0.30, -0.83, -0.47)$	$(0.25, 0.27, -0.32)$
H	$(-0.35, 0.72, 0.59)$	$(-0.79, -0.75, 0.44])$

Table 3: The joint axes for the RRP robot

9 The TP Joint Assembly

A TP joint assembly is an RRP chain in which the axes of the revolute joints intersect at a right angles. This assembly of joints is often used as the shoulder of a robot arm. The P joint provides a sliding boom so the system has a reachable workspace that forms a sphere about the center of the T joint. In this section, we impose constraints on the RRP synthesis equations to obtain design equations for the TP assembly, which we find can be designed to reach three arbitrary spatial positions.

The design equations for an RRP chain are $\mathcal{R}_{1i}$, $\mathcal{S}_{1i}$ and $\mathcal{T}_{1i}$, $i = 2, \ldots, n$, where n is the number of task position together with the five constraint equations $\mathcal{C}_j$, $j = 1, \ldots, 5$.

In order to define the TP chain, we impose three more constraint equations

$$\begin{aligned}
\mathcal{P}_1: &\quad \mathbf{G}\cdot\mathbf{W} = 0,\\
\mathcal{P}_2: &\quad \mathbf{G}\cdot\mathbf{W}^\circ + \mathbf{G}^\circ\cdot\mathbf{W} = 0,\\
\mathcal{P}_3: &\quad \mathbf{m}\cdot\mathbf{H} = 0.
\end{aligned} \tag{44}$$

The first two constraints define the T joint. They ensure that the axes $\mathbf{G}$ and $\mathbf{W}$ are perpendicular and intersect, respectively. A third condition is imposed to obtain same number of equations as unknowns in a situation similar to the RRP chain. In this case, we impose a condition in the components of the slider $\mathbf{H}$.

The unknown parameters in these design equations are the components of the Plücker vectors G and W, and the direction $\mathbf{H}$, for a total of 15 unknowns. The eight constraint equations combine with six design equations for three relative positions (n=3) to yield a total of 14 equations. Thus, one parameter is free to be selected arbitrarily. The structure of these equations is identical to those provided above for the design of the RRP chain.

In order to solve for the directions $\mathbf{G}$ and $\mathbf{W}$, use the design equations $\mathcal{R}_{12}$, $\mathcal{R}_{13}$ and the constraint $\mathcal{P}_1$ to obtain three quadratic equations in six unknowns. Add to this the linear constraints $\mathcal{C}_1$ and $\mathcal{C}_2$ that set the magnitude of these vectors, then all that is needed is that we select the value for one of the parameters of either $\mathbf{G}$ or $\mathbf{W}$. The result is a set of equations that is solved exactly like (33).

Given values for $\mathbf{G}$ and $\mathbf{W}$, we now solve the four design equations $\mathcal{S}_{12}$, $\mathcal{S}_{13}$, $\mathcal{T}_{12}$, $\mathcal{T}_{13}$ and two constraint equations $\mathcal{P}_2$ and $\mathcal{P}_3$ to determine $\mathbf{G}^\circ$, $\mathbf{W}^\circ$ and $\mathbf{H}$. As above we include the constraint equations $\mathcal{C}_3$, $\mathcal{C}_4$, and $\mathcal{C}_5$. A elimination scheme essentially identical to the one used to solve (38) yields solutions for the TP joint assemblies that reach the three specified positions.

10 Example

For this example we use the extra constraints $g_x = 0$ and $h_z = 0$ to solve for the three task positions of Table 4.

We obtain 6 real solutions. Notice that there are only two different solutions for the structural variables that affect the translation only. Figure 7 and Table 5 show one of

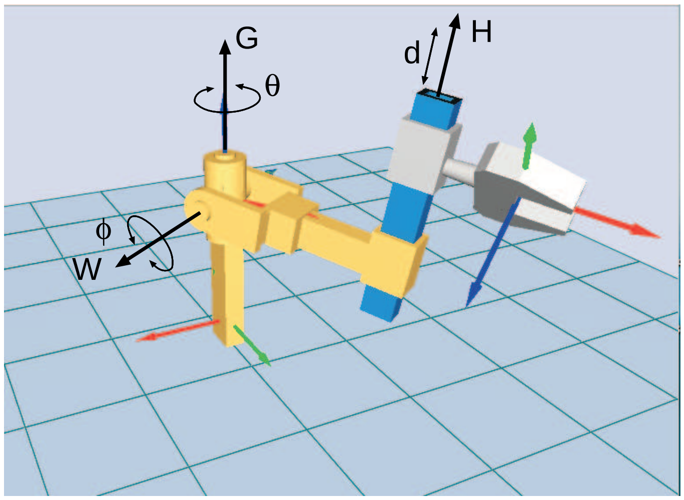

Figure 6: The TP chain is an RRP chain that has revolute axes G and W that are perpendicular and interect.

Table 4: Task positions for a TP chain

Position	*Axis*	*Rotation*	*Translation*
position 1	$(1.0, 0.0, 0.0) + \epsilon(0.0, 0.0, 0.0)$	0^o	0
position 2	$(-0.39, 0.10, 0.92), +\epsilon(-1.73, -3.03, -0.40)$	59.5^o	1.79
position 3	$(-0.38, 0.48, -0.79) + \epsilon(0.72, 0.24, -0.20)$	152.9^o	-2.61

the solutions reaching the task positions.

Table 5: A solution for the TP chain

Joint Axis	*Direction*	*Moment*
G_1	$(0, -0.67, 0.74)$	$(5.01, -1.11, -1.01)$
G_2	$(-0.17, 0.73, 0.66)$	$(-6.83, -2.45, 0.92)$
H	$(0, 0.66, 0.75)$	$(-6.36, 0.36, -0.31)$

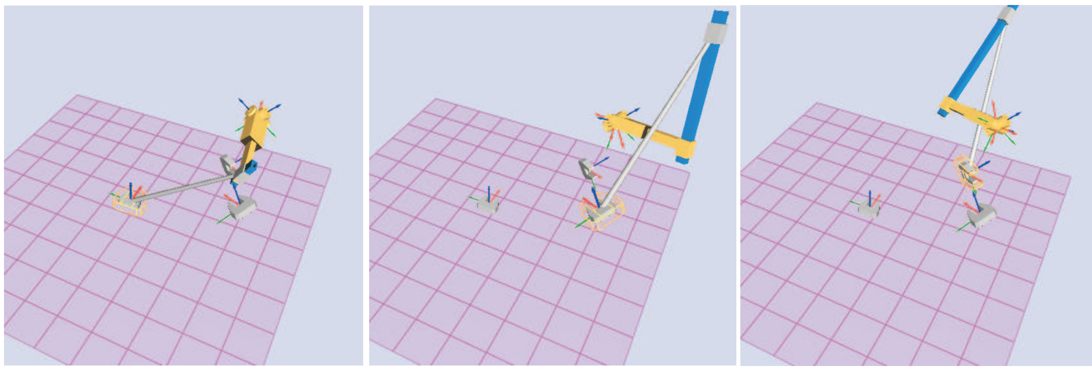

Figure 7: A solution for the TP robot reaching three positions

11 Conclusions

This paper formulates and solves the geometric design problem for RRP, RPR and PRR serial chains using the dual quaternion form of the kinematics equations. The goal is to determine the dimensional parameters of these chains that fit the workspace to four specified task positions. The dual quaternion formalism results in a set of equations that can be collected as matrix equations and inverted formally to eliminate the joint variables of the kinematics equations. The result is a set of polynomial equations in which the dimensional variables can be eliminated explicitly to obtain a univariate polynomial.

The dual quaternion formulation yields equations that are essentially identical for the RRP, RPR and PRR chains, and are solved in the same way. We also present the special case of the TP joint assembly which is a RRP chain in which the axes of the revolute joints are perpendicular and intersect. In this case, the additional constraints result in the ability to design the TP assembly for three arbitrary task positions. The resulting equations have the same structure as in the general problem, and are solved in the same way. We obtain find that there is a maximum of twelve solutions for the four-position synthesis of the RRP chain, and for the three-position synthesis of the TP chain.

We demonstrate this theory by presenting examples of the design of an RRP chain and a TP joint assembly.

12 Acknowledgments

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