Geometric Design of Mechanically Reachable Surfaces

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Overview

• Robot manipulators and spatial mechanisms as assemblies of serial chains.

• *The seven serial chains with reachable surfaces.*

• The model problem: determining a sphere through four points.

• Counting the solutions for a set of polynomial equations.

• *The design equations for the seven serial chains.*

• The homotopy solution methodology for polynomial equations.

• *POLSYS-GLP results.*

• Other serial chains and our Synthetica software.

• Conclusions.
Geometric Design

- The *kinematic structure* of a robot manipulator or spatial mechanism is the network of links and joints that define its movement.
- The primary component of this network is the *serial chain* of links connected by joints. One end of the chain is the base, and the other is the end-effector.

<table>
<thead>
<tr>
<th>Joint</th>
<th>Diagram</th>
<th>Symbol</th>
<th>DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revolute</td>
<td></td>
<td>R</td>
<td>1</td>
</tr>
<tr>
<td>Prismatic</td>
<td></td>
<td>P</td>
<td>1</td>
</tr>
<tr>
<td>Cylindrical</td>
<td></td>
<td>C</td>
<td>2</td>
</tr>
<tr>
<td>Universal</td>
<td></td>
<td>T</td>
<td>2</td>
</tr>
<tr>
<td>Spherical</td>
<td></td>
<td>S</td>
<td>3</td>
</tr>
</tbody>
</table>

The goal of *geometric design* is to determine the physical dimensions of a serial chain that guarantee its end-effector can achieve a specified task.

Serial Robot PUMA 560          Parallel Robot HEXA

UCIrvine
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Reachable Surfaces

PPS chain: Plane

TS chain: Sphere

RPS chain: Hyperboloid

PRS chain: Elliptic cylinder

CS chain: Circular cylinder

right RRS chain: Circular torus

RRS chain: Torus
The Model Problem

• Consider the problem of determining a sphere that contains four specified points.

• Let $R$ be the radius of the sphere, $B=(u, v, w)$ its center, and $P=(X, Y, Z)$ be a general point on the sphere, then we have

$$(X-u)^2 + (Y-v)^2 + (Z-w)^2 = R^2, \quad \text{or} \quad (P-B)\cdot(P-B) = R^2$$

Now let $P^i=(X^i, Y^i, Z^i), i=1, 2, 3, 4$ be four specified points, so we have

$P^i \cdot P^i - 2 P^i \cdot B + B \cdot B = R^2, \quad i=1, 2, 3, 4$

Subtract the first equation from the remaining to cancel the terms $B \cdot B$ and $R^2$, 

$$S: \quad (P^{i+1} \cdot P^{i+1} - P^i \cdot P^i) - 2 (P^{i+1} - P^i) \cdot B = 0, \quad i=1, 2, 3.$$  

This is a set of three linear equations in the parameters $B=(u, v, w)$. There can be at most one solution, which defines the sphere that contains the four points.
Generalizing the Problem

• Rather than specify the points $P^i = (X^i, Y^i, Z^i)$, we now specify seven spatial displacements $[T^i] = [A_i, d_i]$, that define task positions for an end-effector.

• We now seek $p = (x, y, z), B = (u, v, w)$ and $R$, such that $P^i = [T^i]p$ lie on the sphere:

$$ (X^i-u)^2 + (Y^i-v)^2 + (Z^i-w)^2 = R^2, \text{ or } ([T^i]p-B) \cdot ([T^i]p-B) = R^2, \text{ for } i=1, \ldots, 7 $$

Subtract the first equation from the remaining to cancel the terms $B \cdot B$ and $R^2$,

$$ S: \quad (P^{i+1} \cdot P^{i+1} - P^i \cdot P^i) - 2 (P^{i+1} - P^i) \cdot B = 0, \text{ for } i=1, \ldots, 6, \text{ where } P^i = [T^i]p. $$

This is a set of six quadratic equations in the parameters $B = (u, v, w)$ and $p = (x, y, z)$. There can be at most $2^6 = 64$ solutions.

Each solution defines a TS chain that guides its end-effector through the task positions $[T^i]$. 
Counting Solutions

• Total degree: A system of $n$ polynomials of degree $d_1, d_2, \ldots, d_n$ in $n$ variables has at most $D = d_1 d_2 \ldots d_n$ isolated solutions. (Bezout, 1779)


• Multihomogeneous root count: Morgan & Sommese, 1987

General Linear Product

- A polynomial system has the same number, dimension, and degree of solution components for “almost all” values of the coefficients.
- This means we can count solutions (roots) using a polynomial system that has the same monomial structure though different coefficients.

Let \(<u, v, w>\) denote the linear combination \(a_1 u + a_2 v + a_3 w + a_4\) where \(a_i\) are generic coefficients.

Then the quadratic curve \(Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\) can be written as the linear product \(<x, y><x, y> = 0\)

Example: Consider the two plane curves:

\[C_1: A_1 x^2 + B_1 xy + D_1 x + E_1 y + F_1 = 0, \quad <x><x, y>|_1 = 0\]
\[C_2: A_2 x^2 + B_2 xy + D_2 x + E_2 y + F_1 = 0, \quad <x><x, y>|_2 = 0\]

How many solutions? Three.
The equations for the TS chain have the monomial structure:

\[ S: \quad <u, v, w|x, y, z|_i = 0, \quad i = 1, \ldots, 6 \]

Expanding \((P^{i+1} \cdot P^{i+1} - P^i \cdot P^i)\), we find that the quadratic terms cancel and this term actually has the form \(<u, v, w>, which means the TS chain equations actually have the form:

\[ S: \quad <u, v, w|x, y, z|_1 = 0, \]
\[ <u, v, w|x, y, z|_2 = 0, \]
\[ <u, v, w|x, y, z|_3 = 0, \]
\[ <u, v, w|x, y, z|_4 = 0, \]
\[ <u, v, w|x, y, z|_5 = 0, \]
\[ <u, v, w|x, y, z|_6 = 0. \]

And so on for \(\binom{6}{3}\).

Polynomials with this structure have \(\binom{6}{3} = 20\) roots, which means there are at most 20 TS chains that can reach the seven task positions.
## Design Equations

<table>
<thead>
<tr>
<th></th>
<th>Number Task Positions</th>
<th>Equations-degree</th>
<th>Total Degree</th>
<th>GLP Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plane (PPS)</td>
<td>6</td>
<td>5-2nd</td>
<td>32</td>
<td>10</td>
</tr>
<tr>
<td>Sphere (TS)</td>
<td>7</td>
<td>6-2nd</td>
<td>64</td>
<td>20</td>
</tr>
<tr>
<td>Circular Cylinder (CS)</td>
<td>8</td>
<td>7-4th</td>
<td>16,384</td>
<td>2,184</td>
</tr>
<tr>
<td>Hyperboloid (RPS)</td>
<td>10</td>
<td>9-4th</td>
<td>262,144</td>
<td>9,216</td>
</tr>
<tr>
<td>Elliptic Cylinder (PRS)</td>
<td>10</td>
<td>2-3rd, 9-4th</td>
<td>2,097,152</td>
<td>247,968</td>
</tr>
<tr>
<td>Circular Torus (right RRS)</td>
<td>10</td>
<td>1-2nd, 10-4th</td>
<td>2,097,152</td>
<td>868,352</td>
</tr>
<tr>
<td>Torus (RRS)</td>
<td>12</td>
<td>11-4th</td>
<td>4,194,304</td>
<td>448,702</td>
</tr>
</tbody>
</table>

GLP Bound
Solving the Design Equations

• The goal is to find all of the real solutions to the design equations. They are all candidates designs.

• Resultant techniques can be used systems with as many as 50 roots, and eigenvalue elimination methods can extend this to as high as 100 roots.

• Systems of equations with hundreds and thousands of roots require polynomial homotopy solution methods.
Polynomial Homotopy

• Let \( P(\mathbf{z}) \) be the system of polynomial design equations, and we seek all the solutions \( \mathbf{z} \) to \( P(\mathbf{z})=0 \).

• Now let \( Q(\mathbf{z}) \) be a polynomial system that has the same monomial structure as \( P(\mathbf{z}) \), which means we require it to have the same GLP structure.

• The \( N_{GLP} \) roots of \( Q(\mathbf{z})=0 \) are easily computed by solving linear equations.

• Construct the convex combination homotopy \( H(\lambda, \mathbf{z}) = (1- \lambda)Q(\mathbf{z}) + \lambda P(\mathbf{z}) \), where \( \lambda \in [0, 1) \).

• For each root \( \mathbf{z} = \mathbf{a}_j \) of \( Q(\mathbf{z})=0 \) the homotopy equation \( H(\lambda, \mathbf{z}) = 0 \) defines a zero curve \( \gamma_j, j=1, \ldots, N_{GLP} \), which is a connected component of \( H^{-1}(0) \).

Each zero curve of \( H(\lambda, \mathbf{z}) = 0 \) leads either to a root of \( P(\mathbf{z})=0 \) or a root at infinity.
A zero curve can be parameterized by its arc-length $s$, so it has the form $\gamma_j = (\lambda(s), z(s))$. We seek the sequence of points $y_i \approx (\lambda(s_i), z(s_i))$ along $\gamma_j$.

Along the zero curve $\gamma_j$, we have $H(\lambda(s), z(s))=0$, therefore

$$\frac{d}{ds}H(\lambda, z) = \begin{bmatrix} H_\lambda & H_z \end{bmatrix} \begin{bmatrix} d\lambda/ds \\ dz/ds \end{bmatrix} = 0,$$

The matrix $[J_H] = [H_\lambda \ H_z]$ is the nx(n+1) matrix of partial derivatives.

Notice that $v = (d\lambda/ds, dz/ds)$ is tangent to the zero curve, and it is in the null-space of $[J_H]$.

This allows us to estimate the next point along $\gamma_j$ by the formula

$$y_{i+1} = y_i + (s_{i+1} - s_i)v(s_i).$$

This is essentially numerical integration of an ODE and can be solved with efficient predictor-corrector methods. Furthermore, it is well-adapted for parallel computation.
## Number of Solutions

<table>
<thead>
<tr>
<th>Shape</th>
<th>Total Degree</th>
<th>GLP Bound</th>
<th>Number Roots</th>
<th>comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plane (PPS)</td>
<td>32</td>
<td>10</td>
<td>10</td>
<td>resultant</td>
</tr>
<tr>
<td>Sphere (TS)</td>
<td>64</td>
<td>20</td>
<td>20</td>
<td>resultant</td>
</tr>
<tr>
<td>Circular Cylinder (CS)</td>
<td>16,384</td>
<td>2,184</td>
<td>804</td>
<td>PHC 5+ hrs.</td>
</tr>
<tr>
<td>Hyperboloid (RPS)</td>
<td>262,144</td>
<td>9,216</td>
<td>1,024</td>
<td>PHC 24 hrs.</td>
</tr>
<tr>
<td>Elliptic Cylinder (PRS)</td>
<td>2,097,152</td>
<td>247,968</td>
<td>18,120</td>
<td>POLSYS-GLP 30m/8cpu</td>
</tr>
<tr>
<td>Circular Torus (right RRS)</td>
<td>2,097,152</td>
<td>868,352</td>
<td>94,622</td>
<td>POLSYS-GLP 70m/1024cpu</td>
</tr>
<tr>
<td>Torus (RRS)</td>
<td>4,194,304</td>
<td>448,702</td>
<td>42,615</td>
<td>POLSYS-GLP 40m/1024cpu</td>
</tr>
</tbody>
</table>
These results are being integrated into computer-aided design software in the Robotics and Automation Laboratory at UCI.
Other Spatial Chains

Here we have discussed seven spatial serial chains. The spatial constrained serial chains can be enumerated:

<table>
<thead>
<tr>
<th>DOF</th>
<th>Chain</th>
<th>Struct. Params.</th>
<th>Special Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>PP</td>
<td>4</td>
<td>C</td>
</tr>
<tr>
<td></td>
<td>PR</td>
<td>6</td>
<td>T</td>
</tr>
<tr>
<td></td>
<td>RR</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>PPP</td>
<td>6</td>
<td>CP</td>
</tr>
<tr>
<td></td>
<td>PPR</td>
<td>8</td>
<td>CR, PT</td>
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<tr>
<td></td>
<td>PRR</td>
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<td>S, RT</td>
</tr>
<tr>
<td></td>
<td>RRR</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>PPPR</td>
<td>10</td>
<td>CPP</td>
</tr>
<tr>
<td></td>
<td>PPRR</td>
<td>12</td>
<td>CC, CPR, PPT</td>
</tr>
<tr>
<td></td>
<td>PRRR</td>
<td>14</td>
<td>CT, PS, CRR, PRT</td>
</tr>
<tr>
<td></td>
<td>RRRR</td>
<td>16</td>
<td>RS, TT, RRT</td>
</tr>
<tr>
<td>5</td>
<td>PPPRR</td>
<td>14</td>
<td>CCP, CPPR, PPRT</td>
</tr>
<tr>
<td></td>
<td>PPRRR</td>
<td>16</td>
<td>CCR, CPT, PPS, PPRT, CPRR</td>
</tr>
<tr>
<td></td>
<td>PRRRR</td>
<td>18</td>
<td>CS, CRT,PRS, PTT, PRRT</td>
</tr>
<tr>
<td></td>
<td>RRRRR</td>
<td>20</td>
<td>ST, RTT, RPS, RRRT</td>
</tr>
</tbody>
</table>

There are 15 classes with an additional 35 special cases. Including permutations there are 191 chains.
Conclusions

• Kinematic synthesis of spatial chains provides the opportunity to invent new devices for controlled spatial movement.

• The seven serial chains PPS, TS, RPS, PRS, CS, right RRS, and RRS have reachable surfaces that can be shaped so that the end-effector reaches a large number of specified task positions.

• The general cases have a remarkably large number of solutions. Yet 90% of the paths traced by our homotopy algorithm are a waste of cpu-time.

• More efficient solution procedures are needed to make computer-aided-invention practical.

• In fact, we are beginning to consider very large scale computing that evaluates the solutions of the design equations for large number of serial chains.