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SOLVING THE BURMESTER PROBLEM USING KINEMATIC MAPPING

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ABSTRACT

Planar kinematic mapping is applied to the five-position Burmester problem for planar four-bar mechanism synthesis. The problem formulation takes the five distinct rigid body poses directly as inputs to generate five quadratic constraint equations. The five poses are on the fourth order curve of intersection of two hyperboloids of one sheet in the image space. Moreover, the five poses uniquely specify these two hyperboloids. So, given five positions of any reference point on the coupler and five corresponding orientations, we get the fixed revolute centres, the link lengths, crank angles, and the locations of the coupler attachment points by solving a system of five quadratics in five variables that always factor in such a way as to give two pairs of solutions for the five variables (when they exist).

1 Introduction

The determination of a planar four-bar mechanism that can guide a rigid-body through five finitely separated *poses* (position and orientation) is known as the *five-position Burmester problem*, see Burmester (1888). It may be stated as follows. Given five positions of a point on a moving rigid body and the corresponding five orientations of some line on that body, design a four-bar mechanism whose coupler crank pins are located on the moving body and is assemblable upon these five poses. The coupler must assume the five required poses, however sometimes not all five may lie in the same assembly branch.

The problem formulation engenders as many variables as equations so the synthesis is exact. However, most approaches to synthesizing a mechanism that can guide the rigid body exactly through the five positions are rooted in the Euclidean geometry of the plane in which the rigid body must move. From time to time this problem has been revisited (Chang, *et al*, 1991). Readers are refered to this document which contains a recent solution method and a quite adequate and relevant bibliography.

We propose a solution obtained in a three-dimensional projective image space of the rigid body motion. The planar kinematic mapping was introduced independently by Blaschke and Grünwald in 1911 (Blaschke, 1911; Grünwald, 1911). But, their writings are difficult. In North America Roth, De Sa, Ravani (De Sa and Roth, 1981; Ravani and Roth, 1983), as well as others, have made contributions. However, we choose to build upon interpretations by Husty (1995, 1996), who used the accessible language of Bottema and Roth (1990).

Kinematic synthesis of four-bar mechanisms using kinematic mapping was discussed in Bottema and Roth (1990), originally published in 1979, and expanded upon in great detail by Ravani (1982), and Ravani and Roth (1983). In this early work, Ravani and Roth developed the framework for performing *approximate* dimensional synthesis. While *exact* dimensional synthesis for the Burmester problem may have been implied, it has never, to our knowledge, been implemented. Results are so elegantly obtained in the kinematic mapping image space. There-

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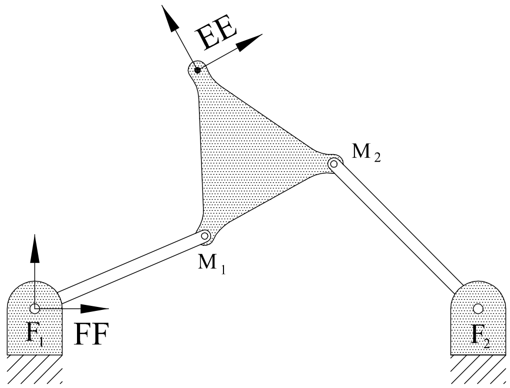

Figure 1. A FOUR-BAR LINKAGE.

fore we are compelled to expose the methodology and procedure by which these are produced.

In this image space, the kinematic constraint implied by the motion of a point bound to move upon a circle of fixed centre and radius maps to a hyperboloid of one sheet. Thus, the motion of the coupler of a planar four-bar mechanism connected with four revolute (R) pairs can be characterized by the fourth order curve of intersection of two distinct hyperboloids of one sheet in the image space.

When the kinematic constraint dictates a point moving on a line with fixed line coordinates, as with a prismatic (P) pair, the constraint surface is a hyperbolic paraboloid. Hyperboloids of one sheet and hyperbolic paraboloids are the only types of constraint surfaces associated with planar mechanisms containing only lower pair joints (Hayes and Husty, 2001). Here, we assume solutions of the five-position Burmester problem confined to four-bar mechanisms jointed with four R-pairs, not slider-cranks. Thus only image space hyperboloids of one sheet will apply.

2 Planar Kinematic Mapping

One can consider the relative displacement of two rigid-bodies in the plane as the displacement of a Cartesian reference coordinate frame EE attached to one of the bodies with respect to a Cartesian reference coordinate frame FF attached to the other. Without loss of generality, FF may be considered as fixed while EE is free to move, as is the case with the four-bar mechanism illustrated by Figure 1. Then the position of a point in EE in terms of the basis of FF can be expressed compactly as

$$\mathbf{p}' = \mathbf{R}\mathbf{p} + \mathbf{d}, \tag{1}$$

where, $\mathbf{p}$ is the 2{ 1 position vector of a point in EE, $\mathbf{p}'$ is the position vector of the same point in FF, $\mathbf{d}$ is the position vector of the origin of frame E in FF, and $\mathbf{R}$ is a 2{ 2 proper orthogonal rotation matrix (*i.e.*, its determinant is $+1$) defined by the orientation of E in FF indicated by ϕ.

Equation (1) can always be represented as a linear transformation by making it *homogeneous* (see McCarthy (1990), for example). Let the homogeneous coordinates of points in the fixed frame FF be the ratios $[X : Y : Z]$, and those of points in the moving frame EE be the ratios $[x : y : z]$. Then Equation (1) can be rewritten as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & a \\ \sin\phi & \cos\phi & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2}$$

2.1 Planar Image Coordinates and Pole Position

The kinematic mapping image coordinates are defined, with respect to P, the pole of a general displacement in the plane shown in Fig. 2, as follows.

$$\begin{aligned} X_1 &= a\sin\frac{\phi}{2} - b\cos\frac{\phi}{2} \\ X_2 &= a\cos\frac{\phi}{2} + b\sin\frac{\phi}{2} \\ X_3 &= 2\sin\frac{\phi}{2} \\ X_4 &= 2\cos\frac{\phi}{2}. \end{aligned} \tag{3}$$

The pole position is invariant with respect to EE and FF, *i.e.*, $X_P = x_P$ and $Y_P = y_P$. Thus using homogeneous coordinates $P\{X_P : Y_P : Z_P\} \equiv P\{x_P : y_P : z_P\}$. This means

$$P\{-b\sin\phi - a(\cos\phi - 1) : -b(\cos\phi - 1) + a\sin\phi : -2(\cos\phi - 1)\}$$

Recall

$$\cos\phi = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}, \;\; \sin\phi = 2\cos\frac{\phi}{2}\sin\frac{\phi}{2}$$
$$(\cos\phi - 1) = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} - \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = -2\sin^2\frac{\phi}{2}.$$

Then

$$P\left\{-2\frac{X_2X_3 - X_1X_4}{X_3^2 + X_4^2}2\cos\frac{\phi}{2}\sin\frac{\phi}{2} - 2\frac{X_1X_3 + X_2X_4}{X_3^2 + X_4^2}(-2\sin^2\frac{\phi}{2}) \right.$$
$$\left. : -2\frac{X_2X_3 - X_1X_4}{X_3^2 + X_4^2}(-2\sin^2\frac{\phi}{2}) + 2\frac{X_1X_3 + X_2X_4}{X_3^2 + X_4^2}2\cos\frac{\phi}{2}\sin\frac{\phi}{2} : -2(-2\sin^2\frac{\phi}{2}\right\}.$$

Dividing by $4\cos\sin\frac{\phi}{2}$ produces

$$P\left\{-\frac{X_2X_3 - X_1X_4}{X_3^2 + X_4^2} + \frac{X_1X_3 - X_2X_4}{X_3^2 + X_4^2}\tan\frac{\phi}{2} : \frac{X_2X_3 - X_1X_4}{X_3^2 + X_4^2}\tan\frac{\phi}{2} + \frac{X_1X_3 + X_2X_4}{X_3^2 + X_4^2} : \tan\frac{\phi}{2}\right\}.$$

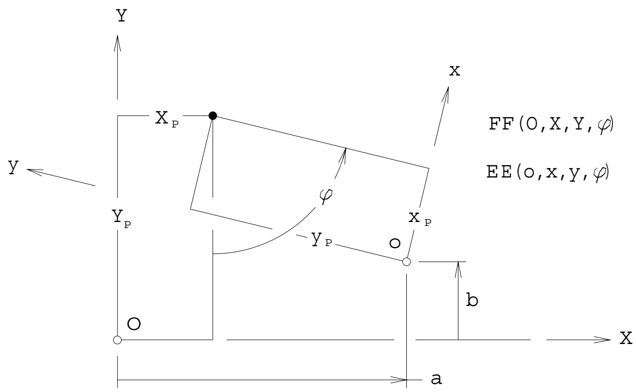

Figure 2. POLE POSITION.

Since

$$\tan\frac{\phi}{2} = \frac{X_3}{X_4}, \tag{4}$$

$$P\left\{\frac{X_1X_3 + X_2X_4 - (X_2X_3 - X_1X_4)\frac{X_4}{X_3}}{X_3^2 + X_4^2} : \frac{X_2X_3 - X_1X_4 + (X_1X_3 + X_2X_4)\frac{X_4}{X_3}}{X_3^2 + X_4^2} : 1\right\}.$$

Which simplifies to

$$P\left\{\frac{X_1}{X_3} : \frac{X_2}{X_3} : 1\right\}, \tag{5}$$

or homogeneously

$$P\{X_1 : X_2 : X_3\}.$$

Armed with Equations (4) and (5) any displacement in terms of X_1, X_2, X_3, X_4 can be conveniently converted to the displacement of EE in terms of FF.

2.2 Representing Planar Displacements in Terms of Image Space Coordinates

As regards the plane, recall the general homogeneous transformation, Equation (2), of a point $\{x : y : z\}$ in the moving, end effector EE frame measured as $\{X : Y : Z\}$ in the fixed reference frame FF expressed in terms of the kinematic mapping image space coordinates as extracted from the above development.

$$\lambda \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{6}$$

The inverse transformation can be obtained with the inverse of the 3{ 3 matrix in Equation (6) as follows.

$$\lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & 2X_3X_4 & 2(X_1X_3 - X_2X_4) \\ -2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 + X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \tag{7}$$

where the product of these matrices is not a unit matrix but yields

$$\begin{bmatrix} (X_3^2 + X_4^2)^2 & 0 & 0 \\ 0 & (X_3^2 + X_4^2)^2 & 0 \\ 0 & 0 & (X_3^2 + X_4^2)^2 \end{bmatrix}.$$

However this is a diagonal matrix with identical elements and homogeneous coordinates may be multiplied by any value $X_3^2 + X_4^2 \neq 0$ while preserving the unique point they represent. This argument also provides the necessary non-zero condition. Furthermore λ, above and below, is some non-zero constant. Clearly, by construction in Equation (3), $X_3^2 + X_4^2 \equiv 2$.

2.3 Planar Constraint Equations

Consider the case of an R-R joint dyad. A point on EE moves on a circle on FF. Substituting the equation $X_i = X_i(x, y, z)$ implied by Equation (6) into the circle equation

$$C_0(X^2 + Y^2) + 2C_1X + 2C_2Y + C_3 = 0 \tag{8}$$

produces a hyperboloid of one sheet in the image space, see Figure 3. In Equation (8) $C_0 = k$, an arbitrary constant, while $C_1 = -X_m$, $C_2 = -Y_m$, the circle centre coordinates, and $C_3 = X_m^2 + Y_m^2 - r^2$ with r being the circle radius. The hyperboloid takes the form:

$$\begin{aligned} C_0z^2(X_1^2 + X_2^2) + (-C_0x + C_1z)zX_1X_3 \\ +(-C_0y + C_2z)zX_2X_3 + (-C_0y - C_2z)zX_1X_4 \\ +(C_0x + C_1z)zX_2X_4 + (-C_1y + C_2x)zX_3X_4 \\ +\frac{1}{4}[C_0(x^2 + y^2) - 2C_1xz - 2C_2yz + C_3z^2]X_3^2 \\ +\frac{1}{4}[C_0(x^2 + y^2) + 2C_1xz + 2C_2yz + C_3z^2]X_4^2 = 0. \end{aligned} \tag{9}$$

Note that setting $C_0 = z = X_4 = 1$ produces

$$\begin{aligned} (X_1^2 + X_2^2) + (C_1 - x)X_1X_3 + (C_2 - y)X_2X_3 \\ \mp(C_2 + y)X_1 \pm (C_1 + x)X_2 \pm (C_2x - C_1y)X_3 \\ +\frac{1}{4}[(x^2 + y^2) - 2C_1x - 2C_2y + C_3]X_3^2 \\ +\frac{1}{4}[(x^2 + y^2) + 2C_1x + 2C_2y + C_3] = 0. \end{aligned} \tag{10}$$

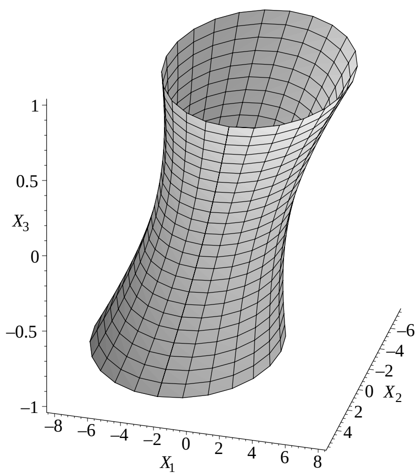

Figure 3. A HYPERBOLOID OF ONE SHEET.

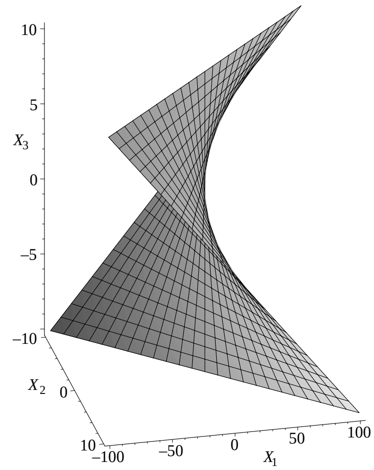

Figure 4. A HYPERBOLIC PARABOLOID.

where (x,y) are the coordinates of the moving point expressed in EE with $z=1$ and the upper signs apply. If the constraint is intended to express the inverse, a point on FF bound to a circle in EE, then the lower signs apply and x, y or z is substituted wherever X, Y or Z appears. The inverse situation of a circle moving on a point is never required in problem formulation.

However if a point is bound to a line, *i.e.*, in the case of a prismatic joint, and if one desires to treat inversions, the line may be either on FF *or* EE. Equation (10) reduces to Equation (11) if a point is bound to a line and $C_0 = 0$. This produces a hyperbolic paraboloid in the image space, see Figure 4:

$$C_1X_1X_3 + C_2X_2X_3 \mp C_2X_1 \pm C_1X_2 \pm (C_2x - C_1y)X_3$$
$$-\frac{1}{4}[2C_1x + 2C_2y - C_3]X_3^2 + \frac{1}{4}[2C_1x + 2C_2y + C_3] = 0. \quad (11)$$

3 The Five-Position Burmester Problem

The goal of the dimensional synthesis problem for rigid body guidance is to find the *moving circle points*, M_1 and M_2 of the coupler, i.e., the revolute centres that move on fixed centred, fixed radii circles as a reference coordinate system, EE, attached to the coupler, passes through the desired poses. The *fixed centre points* for each circle are the fixed, or grounded revolute centres, F_1 and F_2, respectively. The circle and centre points are illustrated with the four-bar mechanism shown in Figure 1. For these constraints, the synthesis equations are determined using Equation (10).

What we set out to do here is to use the methods of planar kinematic mapping outlined in (Zsombor-Murray, *et al*, 2002) and set up five simultaneous constraint equations, each of which represents the image space surface of a rigid body moving freely in the plane except that one point is bound to the circumference of a fixed circle. These equations are expressed in terms of the following eight variables.

i. $X_1,\ X_2,\ X_3$, the dehomogenized coordinates of the coupler pose in the image space.
ii. $C_1,\ C_2,\ C_3$, the coefficients of a circle equation ($C_0 = 1$).
iii. $x,\ y$, the "home" position coordinates of a crank-pin revolute point, on the coupler, which moves on a circle. This is its position displaced from $(a,b) = (0,0)$ and $\phi = 0$ as implied in the five pose given set.

Since $X_1,\ X_2,\ X_3$ are given for five desired coupler poses, one may in principle solve for the remaining five variables. If two real solutions occur then all design information is available.

i. Circle centre is at $X_m = -C_1,\ Y_m = -C_2$.
ii. Circle radius is given by $r^2 = C_3 - (X_m^2 + Y_m^2)$.
iii. Coupler length is given by $L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$; remember there are two circles in a feasible mechanism design result.

4 Analysis

4.1 Crank Angles

If the desired five poses can be realized with a planar 4R four-bar mechanism, then at least two real solutions in $(C_1,\ C_2,\ C_3,\ x,\ y)$ will be obtained, defining two dyads sharing the coupler. To construct the mechanism in its five configurations the crank angles must be determined. To obtain the crank angles one just takes (x_1,y_1) and (x_2,y_2) and performs the linear transformation, expressed in image space coordinates, five times.

$$\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1-X_3^2 & -2X3 & 2(X_1X_3 + X2) \\ 2X_3 & 1-X_3^2 & 2(X_2X_3 - X_1) \\ 0 & 0 & 1+X_3^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

(X,Y) come in five pairs because there were five poses given *a-priori*. These are the Cartesian coordinates of the moving revolute centres expressed in FF, and implicitly define the crank angles. For a practical design one must check that the solution did not separate crank pin coordinates in unconnected mechanism branches.

4.2 Converting Pose to Image Space Coordinates

This is the first step in the analysis so in reality we started at the end and now jump to the beginning of the procedure. The sequence was chosen in the hope that it might facilitate understanding of the algorithm, if not its theoretical background which for our purposes is avoided here. Now examine Equations (3) and divide by X_4.

$$
\begin{aligned}
X_1 &= \frac{\left(a\tan\frac{\phi}{2} - b)\right)}{2}\\
X_2 &= \frac{\left(a + b\tan\frac{\phi}{2}\right)}{2}\\
X_3 &= \tan\frac{\phi}{2}\\
X_4 &= 1.
\end{aligned}
$$

The five given poses being specified as (a,b,ϕ), the planar coordinates of the moving point and the orientation of a line on the moving rigid body, all with respect to $(0,0,0°)$ expressed in FF. Note that the location of the origin of FF is arbitrary, it is only shown on the fixed revolute centre in Figure 1 for convenience.

4.3 Pose Constraint Equation

Given the constraints imposed by four revolute joints, the pose constraint equation (synthesis equation) is given by Equation (10) with the upper signs used. All variables and parameters have been previously defined, we obtain:

$$
\begin{aligned}
&(X_1^2+X_2^2)+(C_1-x)X_1X_3+(C_2-y)X_2X_3\\
&-(C_2+y)X_1+(C_1+x)X_2+(C_2x-C_1y)X_3\\
&+\tfrac{1}{4}[(x^2+y^2)-2C_1x-2C_2y+C_3]X_3^2\\
&+\tfrac{1}{4}[(x^2+y^2)+2C_1x+2C_2y+C_3]=0.
\end{aligned} \tag{12}
$$

5 Example and Verification

The kinematic mapping solution to the five-position Burmester problem is illustrated with the following example problem. In order to verify our synthesis results, we started with Figure 5, wherein one sees a four-bar mechanism design represented by dotted crank pin circles and a coupler CD which has

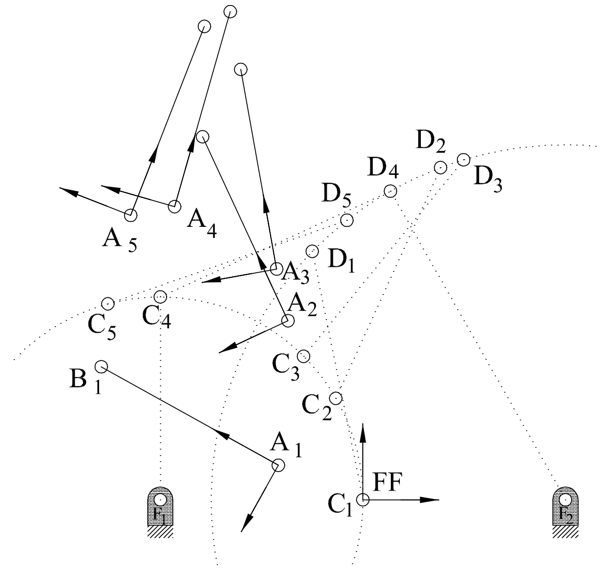

Figure 5. GENERATING THE FIVE DESIRED POSES.

been placed in five feasible poses. Then an arbitrary point A and orientation line AB were specified. These were used to specify the given five poses, listed in Table 2. The fixed revolute centres and link lengths of the four-bar mechanism used to generate the poses, which we can check for verification, are listed in Table 1, all coordinates given relative to FF. The coordinate information obtained from these were inserted into the five synthesis equations. The results at the end constitute obvious confirmation concerning the effectiveness of the kinematic mapping approach to solving the Burmester problem.

Parameter	Value
F_1	(-8,0)
F_2	(8,0)
F_1F_2	16
F_1C	8
CD	10
DF_2	14

Table 1. THE GENERATING MECHANISM

Given the Cartesian coordinates of five positions of a reference point on a rigid body, together with five orientations of the rigid body which correspond to the positions, all relative to an

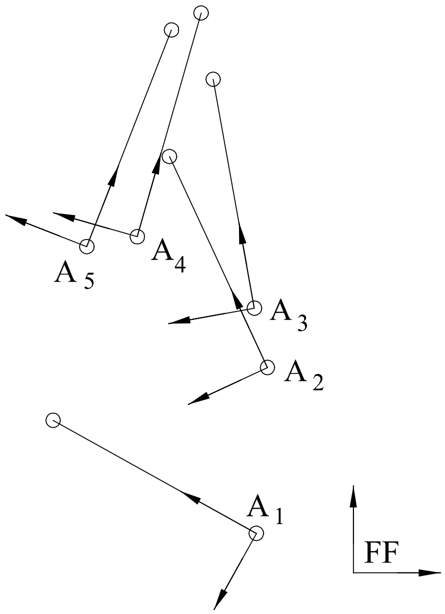

Figure 6. THE FIVE DESIRED POSES.

arbitrary fixed reference frame, FF. The reference point is the origin of a coordinate system, A, attached to the rigid body. In Figure 6 the five poses are indicated by the position of A and the orientation of a line in the direction x_A axis. The coordinates and orientations are listed in Table 2.

i^{th} Pose, A_i	a	b	φ (deg)
1	-3.339	1.360	150.94
2	-2.975	7.063	114.94
3	-3.405	9.102	100.22
4	-7.435	11.561	74.07
5	-9.171	11.219	68.65

Table 2. FIVE RIGID BODY POSES IN FF.

The given five poses are mapped to five sets of coordinates in the image space. Using a computer algebra software package, we substitute the corresponding values for X_1, X_2, X_3, together with $X_4 = 1$ into Equation (12) yields the following five quadratics in C_1, C_2, C_3, x, and y:

$$\begin{aligned} &51.62713350 - 26.52347891C_1 + 28.43187273x \\ &+3.439909575y + 10.80321393C_2 + 3.971769828y^2 \\ &+3.971769828x^2 - 6.943539655C_1x + 3.971769828C_3 - \\ &6.943539655C_2y - 3.858377808C_1y + 3.858377808C_2x = 0 \end{aligned} \tag{13}$$

$$\begin{aligned} &50.78111719 - 5.144112496C_1 + 13.24300208x \\ &-.485305000y + 12.21272826C_2 + .8645567222y^2 \\ &+.8645567222x^2 - .7291134440C_1x + .8645567222C_3 \\ &-.7291134440C_2y - 1.567873365C_1y + 1.567873365C_2x = 0 \end{aligned} \tag{14}$$

$$\begin{aligned} &57.40558942 - 4.139456673C_1 + 11.62418825x \\ &+2.110482435y + 11.06529652C_2 + .6078497318y^2 \\ &+.6078497318x^2 - .2156994635C_1x + .6078497318C_3 \\ &-.2156994635C_2y - 1.196410852C_1y + 1.196410852C_2x = 0 \end{aligned} \tag{15}$$

$$\begin{aligned} &74.12376162 - 5.833830775C_1 + 7.121746695x \\ &+8.099525062y + 9.071273378C_2 + .3923221773y^2 \\ &+.3923221773x^2 + .2153556452C_1x + .3923221773C_3 \\ &+.2153556452C_2y - .7545122328C_1y + .7545122328C_2x = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} &76.96602922 - 6.723290851C_1 + 5.212549019x \\ &+9.256210937y + 8.224686519C_2 + .3665516768y^2 \\ &+.3665516768x^2 + .2668966465C_1x + .3665516768C_3 \\ &+.2668966465C_2y - .6827933120C_1y + .6827933120C_2x = 0 \end{aligned} \tag{17}$$

Solving the system of Equations (13)-(17) yields four sets of values for C_1, C_2, C_3, x, and y, two being real, and the remaining two being complex conjugates. The two real sets of hyperboloid coefficients are listed in Table 3. The corresponding synthesized four-bar fixed revolute centres and link lengths are listed in Table 4, rounded to same three decimal places as the graphically determined generating mechanism listed in Table 1.

Coefficient	Solution 1	Solution 2
C_1	-7.983138944	7.997107716
C_2	-.027859304	-.000953257
C_3	-131.4773813	-.022545268
x	2.932070052	-3.579426217
y	-8.023883728	-.435620093

Table 3. THE HYPEBOLOID COEFFICIENTS

While the synthesized mechanism link lengths and centre coordinates are affected by the numerical resolution of the graphical construction of the generating mechanism, we believe this example demonstrates the utility of kinematic mapping to solving the five-position Burmester problem.

Parameter	Value
F_1	(-7.997,0.001)
F_2	(7.983,-0.023)
F_1F_2	15.980
F_1C	7.999
CD	10.003
DF_2	13.972

Table 4. THE SYNTHESIZED MECHANISM

6 Computational Pathology

Notice that feasible slider-crank solutions were implicitly excluded by choosing to set $C_0 = z = 1$ rather than, say, $C_2 = y = 1$. This is similar to excluding half-turn EE orientations by setting $X_4 = 1$ rather than, say, $X_3 = 1$. It is recommended that algorithmic implementation should retain $X_4 = 2\cos(\phi/2)$ and contain features to replace $C_0 = 1$ with C_1, C_2 or $C_3 = 1$ and $z = 1$ with x or $y = 1$ should results where $x \rightarrow y \rightarrow \infty$ with $C_0 = z = 1$ occur.

7 Conclusion

To quote Bugs Bunny, "That's all, folks!" As Chao Chen (4) observed so astutely in his thesis conclusion, "The author found this tool (kinematic mapping) to be easy to use but difficult to understand. Therefore, it may obtain wider acceptance if we focus first on applications before becoming enmeshed deeply in the theoretical background." Witness the results (14) of Ravani's thesis. He was ever so close to having done just what we have outlined above and even suggested applications to kinematic synthesis, specifically four-bar mechanism guidance problems. Much was revealed concerning the structure of the one-sheet hyperboloid surfaces in the image space and the nature of the curve of intersection between two. Fortunately for us, he overlooked the opportunity to expose a simple algorithmic solution to the Burmester problem.

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